On M5-branes in ABJM theory

Seiji Terashima (YITP)

12 August 2014 at Hannover

mainly based on the papers:

"Integrability of BPS equations in ABJM theory "

K. Sakai and ST, JHEP11(2013)002,

"M5-branes in ABJM theory and Nahm equation"

T.Nosaka and ST, Phys.Rev. D86 (2012) 125027

M-theory is “defined” by

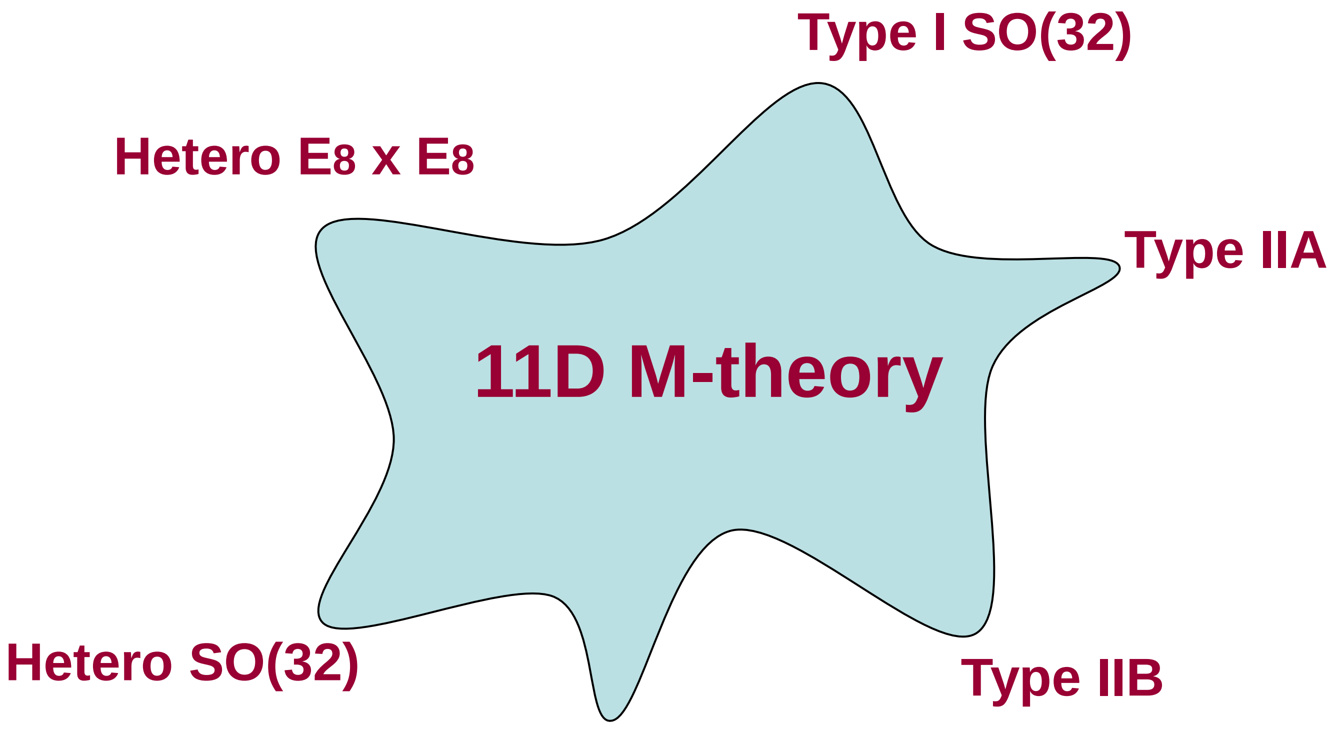
non-perturbative IIA string theory

More precisely,

M-theory on S^1 = type IIA string

radius of S^1 ~ string coupling constant

It was claimed:
"All string theories are unified"



Any string theory = a compactification of M-theory

Various string dualities are manifest in M-theory

M-theory

is described by

11d supergravity
in a low energy limit

Fields in 11d sugra
are
3-forms $C_{\mu\nu\rho}$ and metric only.

(except gravitino)

↓

Only M2-branes and M5-branes
are (charged) branes

Thus,

M2-branes and M5-branes

are

highly important
for understanding M-theory

Branes also are unified into *M*-branes,

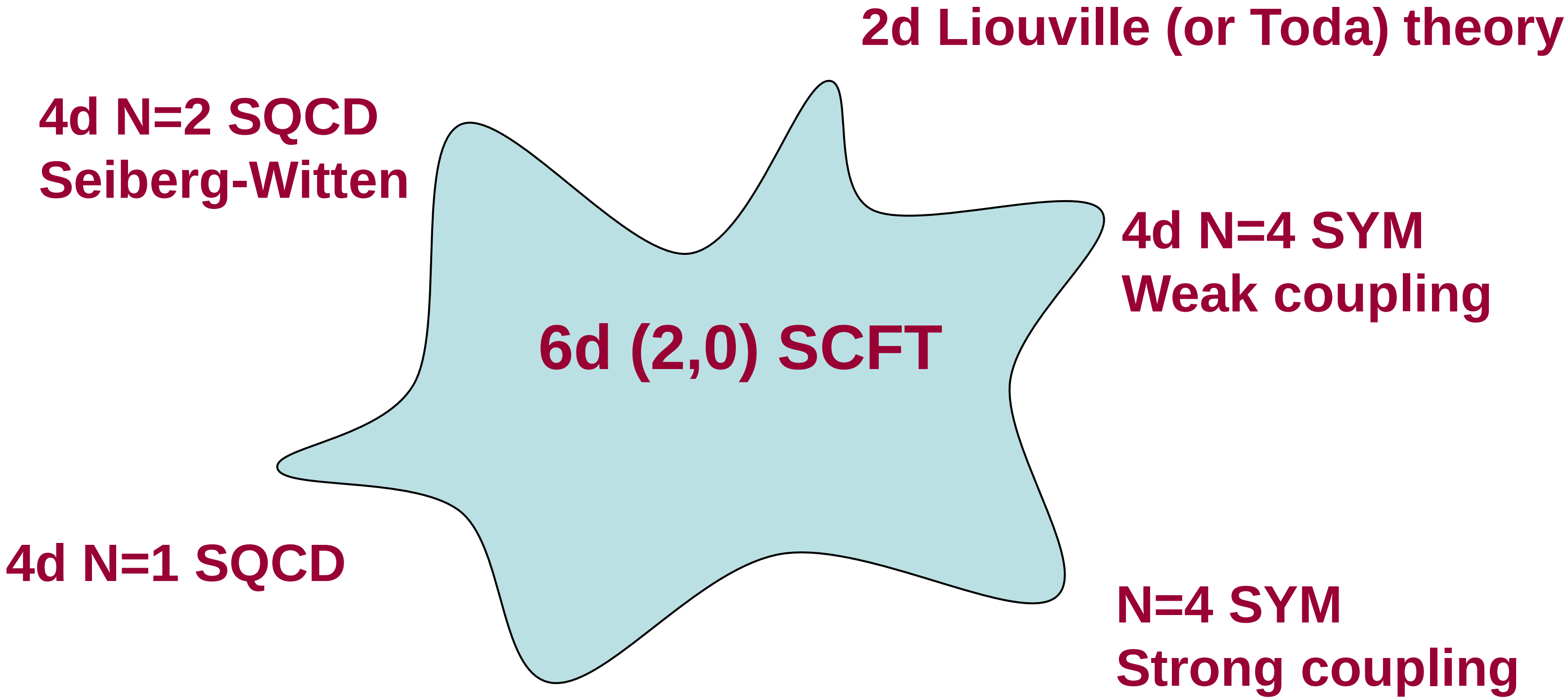
⇓

there is a field theory analogue of the <u>*M*-theory</u> web

⇕

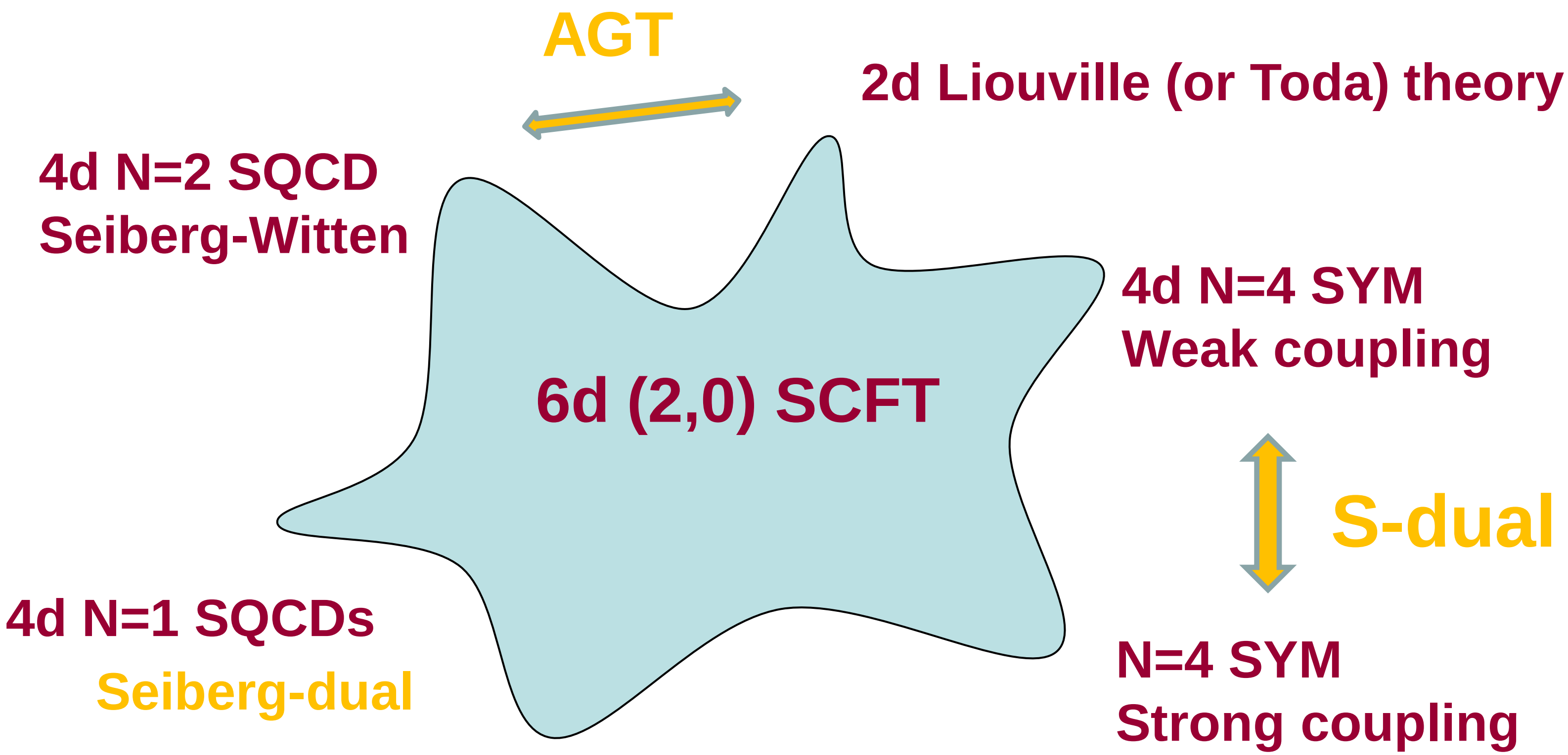
<u>6d (2,0) superconformal field theory</u>
defined by
a low energy effective theory
on *M*5-branes

Field Theory Analogue of M-theory web:



Many 4d field theories = compactifications of the 6d theory

Field Theory Analogue of M-theory web:



Many 4d field theories = compactifications of the 6d theory

6d theory as an origin of various dualities!

Understanding

6d (2,0) SCFT (= LEEA of M5-branes)

is

obviously

interesting and important !

However,

There is no (explicit) definition of 6d (2,0) SCFT….

a critical person would say
that
this 6d (2,0) SCFT is just an imagination

So far, no significant progress for understanding *M5*-branes themselves.

We should try to go any direction!

On the other hand,

the other M-brane, i.e. M2-brane,
has been significantly understood recently.

The low energy effective theory on M2-branes
is given by
ABJM(Aharony-Bergman-Jafferis-Maldacena) theory,
which is just a 3d Chern-Simons-matter theory.

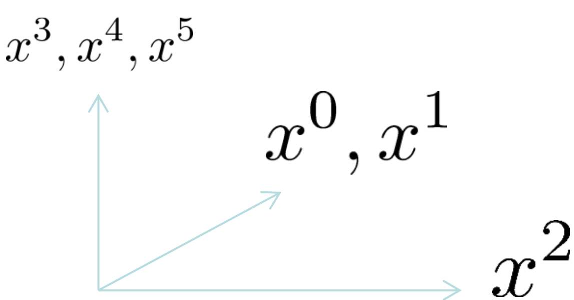
We expect that

the *M2*-branes know the *M5*-branes.

Why?

Now, remember that in type IIA string theory, D2-branes know D4-branes.

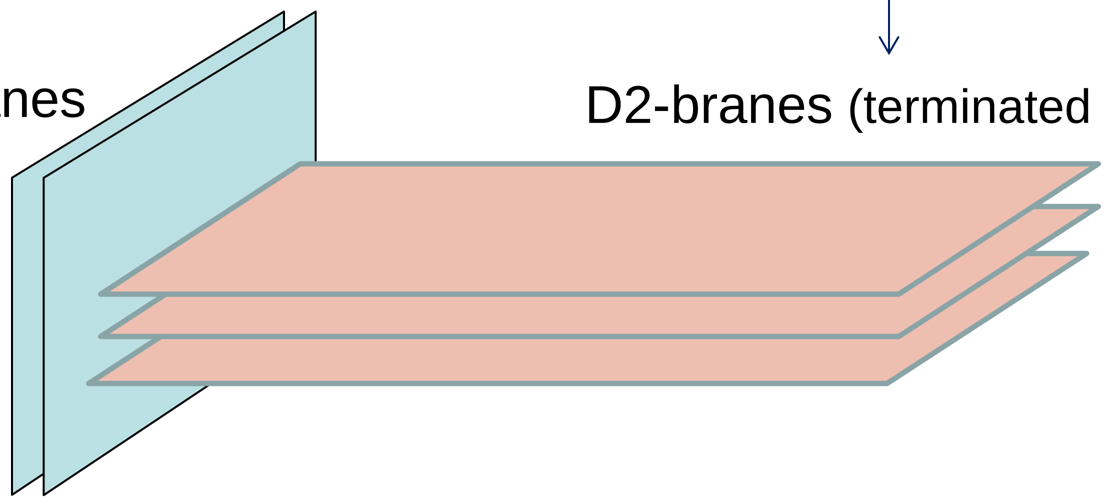
D2-branes (no boundary)

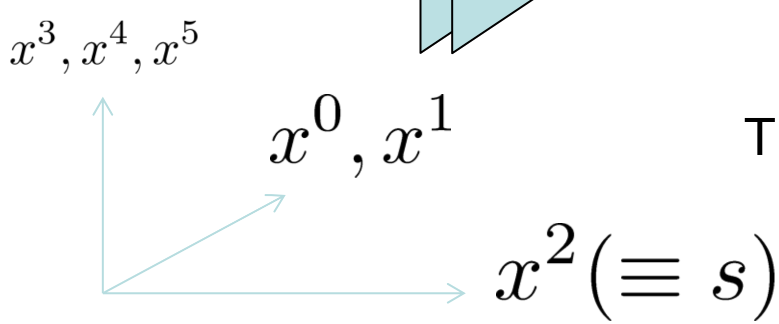


Super Yang-Mills theory describes these D2-branes

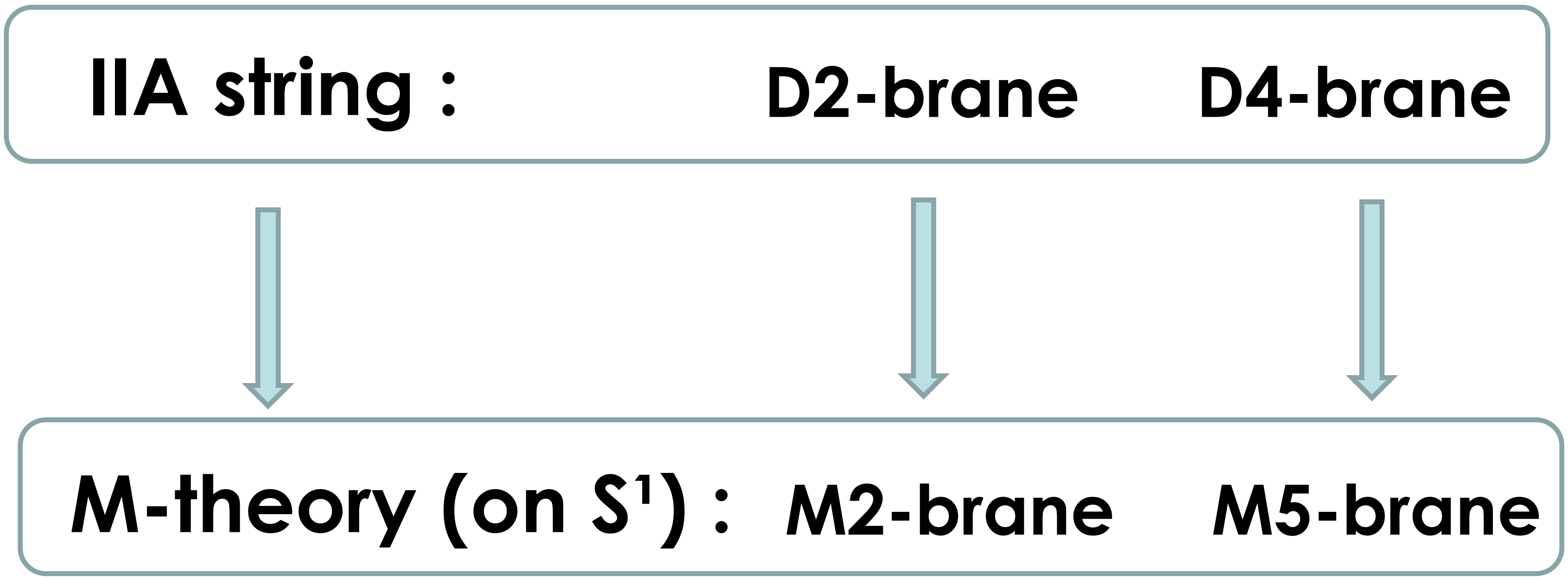
ADHM(N) construction

Monopole equation	⟺	Nahm equation
$\frac{1}{2}\epsilon_{IJK}F^{IJ} = D_K\Phi$		$\dot{T}^I = i\epsilon_{IJK}T^J T^K$
↓		↓
D4-branes		D2-branes (terminated at D4-branes)



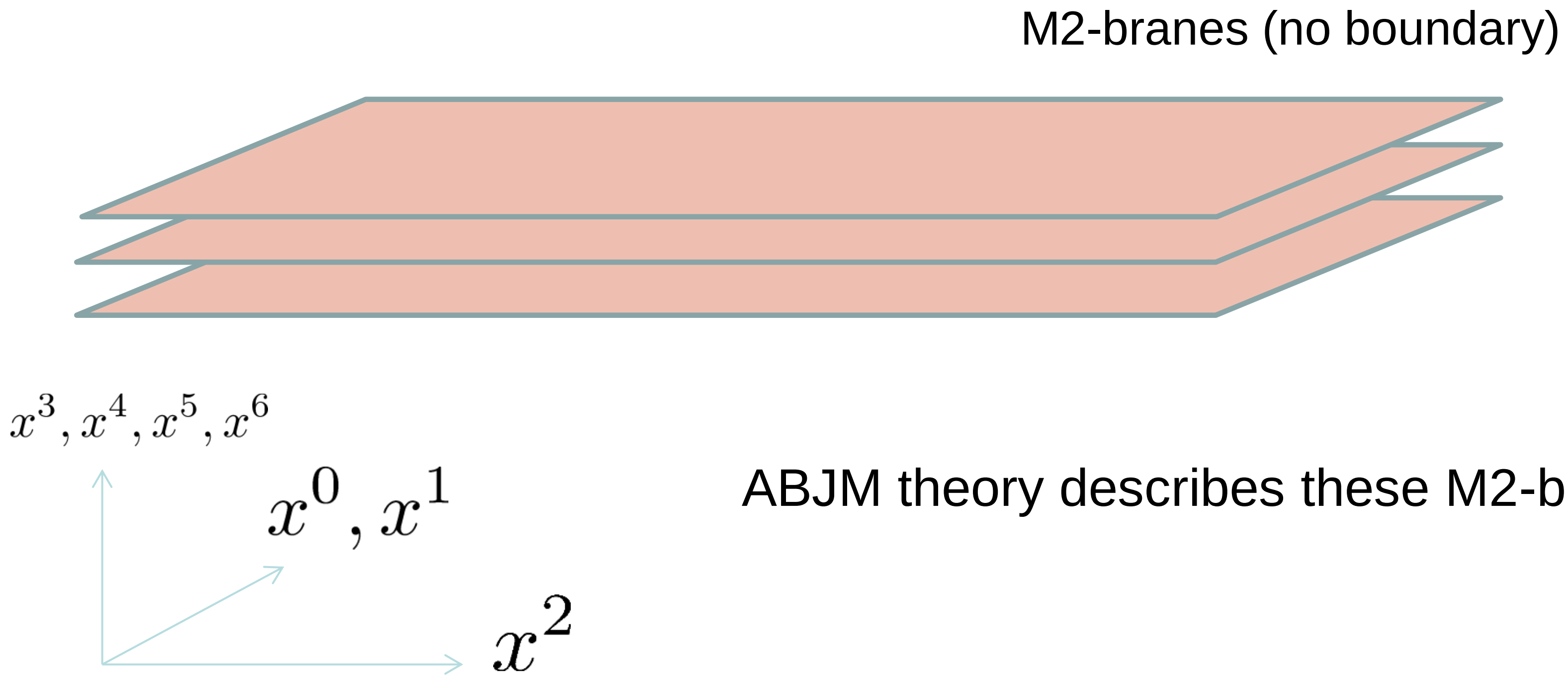


This is a soliton solution in Super YM theory


IIA string :
D2-brane
D4-brane
M-theory (on S^1) :
M2-brane
M5-brane

We expect that

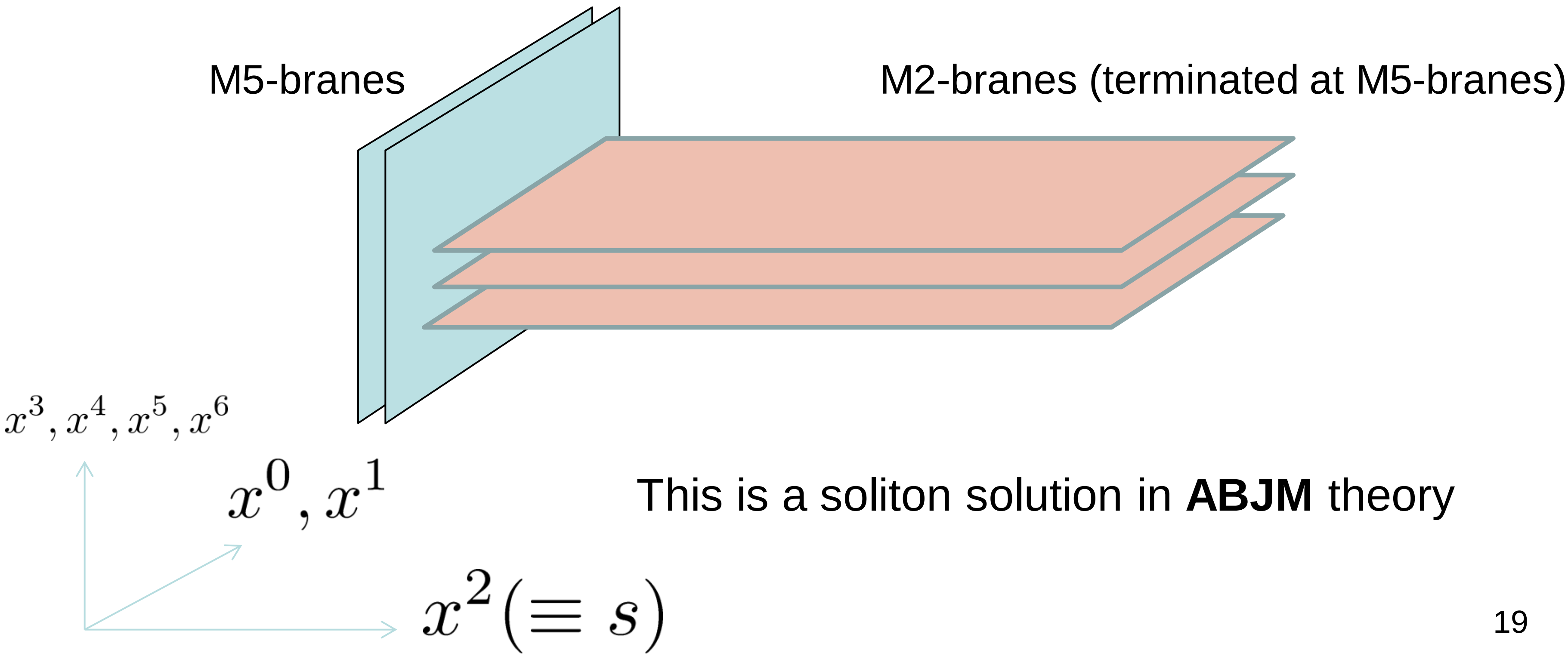
the M2-branes know the M5-branes.



ABJM theory describes these M2-branes

We expect that

the M2-branes know the M5-branes.



This is a soliton solution in **ABJM** theory

In ABJM theory, BPS equations for this M5-brane-M2-brane bound state is

$$0 = \frac{dY^1}{dx^2} + \frac{2\pi}{k}(Y^2 Y_2^\dagger Y^1 - Y^1 Y_2^\dagger Y^2),$$
$$0 = \frac{dY^2}{dx^2} + \frac{2\pi}{k}(Y^1 Y_1^\dagger Y^2 - Y^2 Y_1^\dagger Y^1),$$

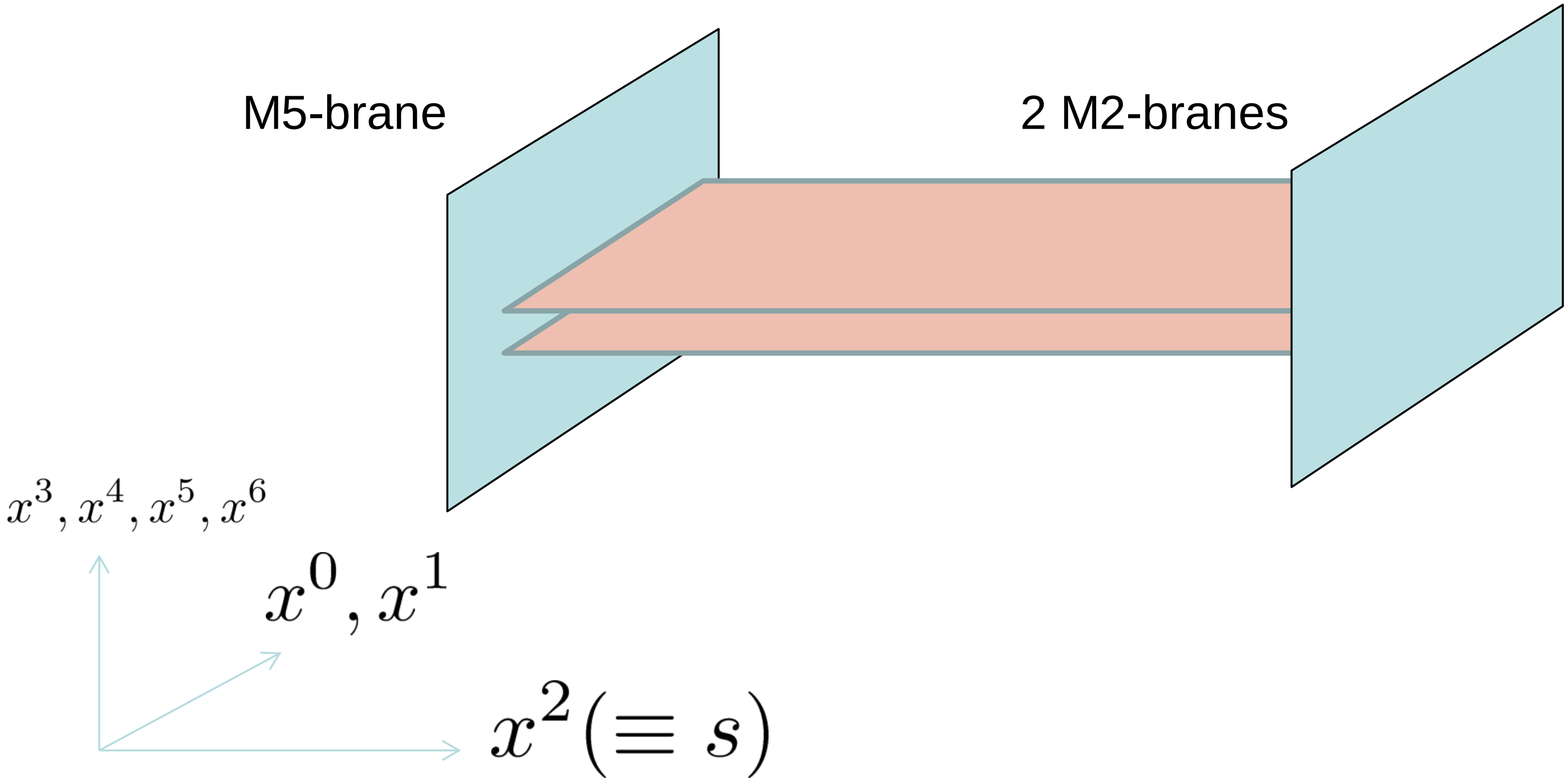
Thus,
the solutions of these BPS eq.
should represent the M5-branes.
Thus those will be important!

(for example, in order to try to find
a Nahm-like transformation to
the BPS solutions in the M5-brane action).

However,

only a few solutions had been known.

Recently, we found **all solutions** for two M2-branes:



The KEY facts:

The BPS equations

$$0 = \frac{dY^1}{dx^2} + \frac{2\pi}{k}(Y^2 Y_2^{\dagger} Y^1 - Y^1 Y_2^{\dagger} Y^2),$$
$$0 = \frac{dY^2}{dx^2} + \frac{2\pi}{k}(Y^1 Y_1^{\dagger} Y^2 - Y^2 Y_1^{\dagger} Y^1),$$

⇕ **equivalent!**

Lax equation $\dot{A} = [A, B]$ for Lax pair

$$A(s;\lambda) = \begin{pmatrix} O & Y^1 + \lambda Y^2 \\ Y^{1\dagger} - \lambda^{-1} Y^{2\dagger} & O \end{pmatrix},$$
$$B(s;\lambda) = \begin{pmatrix} \lambda^{-1} Y^1 Y^{2\dagger} + \lambda Y^2 Y^{1\dagger} & O \\ O & \lambda Y^{1\dagger} Y^2 + \lambda^{-1} Y^{2\dagger} Y^1 \end{pmatrix}$$

Using this integrable structure rather tricky,

we find all solutions for two M2-branes.

This could be a small step toward understanding M5-branes

I will talk about this.

Plan

- **ABJM theory and the BPS equations**
- **The Lax pair**
- **The solutions for two M2-branes**

The ABJM theory and the BPS equations

M2-brane effective action (3 dim. field theory) should have

- 8 scalars = location of M2-brane in 11d spacetime
- 16 SUSY and SO(8) global symmetry
- Conformal symmetry (→ not Yang-Mills theory)

Fields in ABJM action:

4 complex scalars (A=1,2,3,4)
bi-fundamental rep. of *U(N) x U(N)*

$$Y^A , Y_A^\dagger$$

location of M2-branes

4 (2+1)d Dirac spinors
bi-fundamental rep. of *U(N) x U(N)*

$$\psi_A , \psi^{A\dagger}$$

(2+1)d *U(N) x U(N)* gauge fields

$$A_\mu , \hat{A}_\mu$$

ABJM action

is

U(N) x U(N) Chern-Simons-matter theory

which has

- 12 SUSY (N=6)
- SU(4) x U(1) global symmetry
- Conformal symmetry

This action describes *N* M2-branes on $\mathbf{C}^4/\mathbf{Z}_k$

$$Y^A \to e^{2\pi i/k} Y^A$$

ABJM action:

$$S=\int d^3x\left[\frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda+\frac{2i}{3}A_\mu A_\nu A_\lambda-\hat{A}_\mu\partial_\nu\hat{A}_\lambda-\frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right)\right.$$
$$\left.-\mathrm{Tr}D_\mu Y_A^\dagger D^\mu Y^A-i\mathrm{Tr}\,\psi^{A\dagger}\gamma^\mu D_\mu\psi_A-V_{\mathrm{bos}}-V_{\mathrm{ferm}}\right]$$

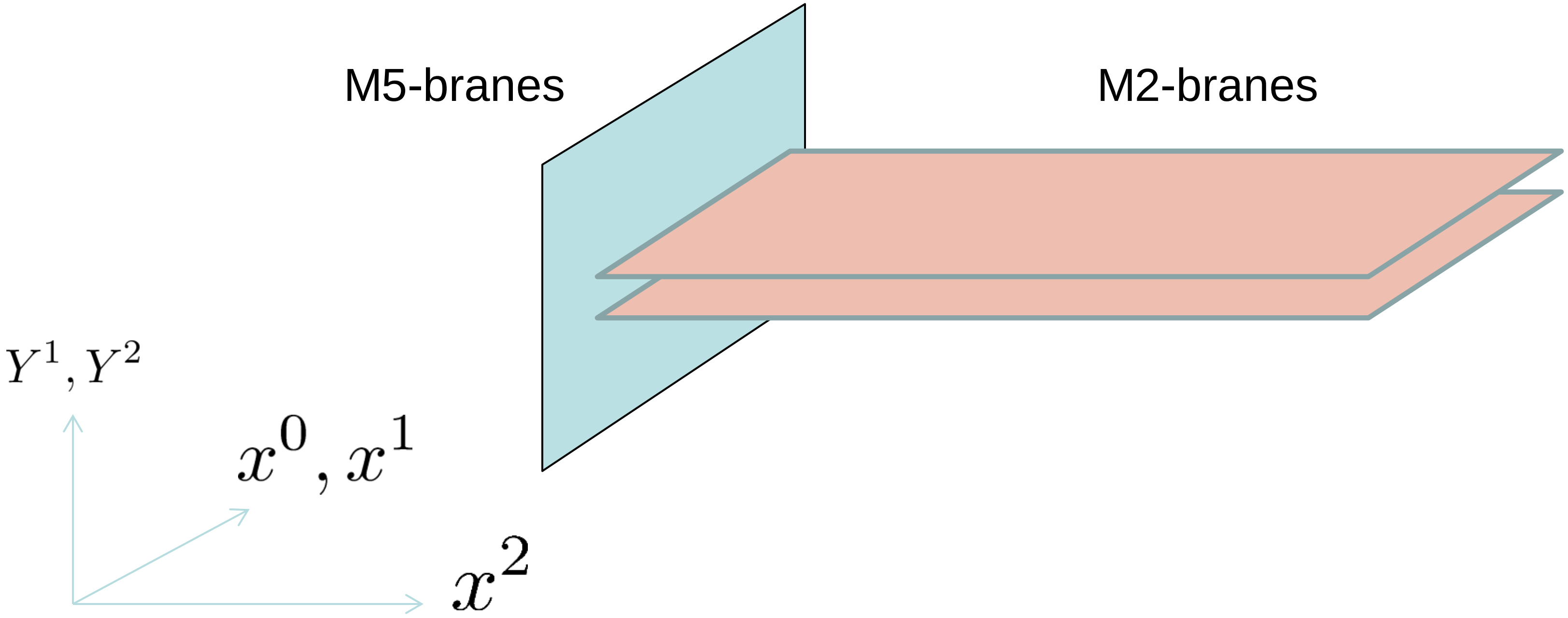
$$V_{bos}=-\frac{4\pi^2}{3k^2}\mathrm{Tr}\left(Y^AY_A^\dagger Y^BY_B^\dagger Y^CY_C^\dagger+Y_A^\dagger Y^AY_B^\dagger Y^BY_C^\dagger Y^C\right.$$
$$\left.+4Y^AY_B^\dagger Y^CY_A^\dagger Y^BY_C^\dagger-6Y^AY_B^\dagger Y^BY_A^\dagger Y^CY_C^\dagger\right)$$

$$V_{ferm}=-\frac{2i\pi}{k}\mathrm{Tr}\left(Y_A^\dagger Y^A\psi^{B\dagger}\psi_B-\psi^{B\dagger}Y^AY_A^\dagger\psi_B-2Y_A^\dagger Y^B\psi^{A\dagger}\psi_B+2\psi^{B\dagger}Y^AY_B^\dagger\psi_A\right.$$
$$\left.+\epsilon^{ABCD}Y_A^\dagger\psi_BY_C^\dagger\psi_D-\epsilon_{ABCD}Y^A\psi^{B\dagger}Y^C\psi^{D\dagger}\right),$$

M2-M5-brane bound state in ABJM

We assume

$$\begin{cases} Y^1 = Y^1(x^2) \\ Y^2 = Y^2(x^2) \\ Y^3 = Y^4 = 0 \\ A_\mu = \hat{A}_\mu = 0 \end{cases}$$



Then,

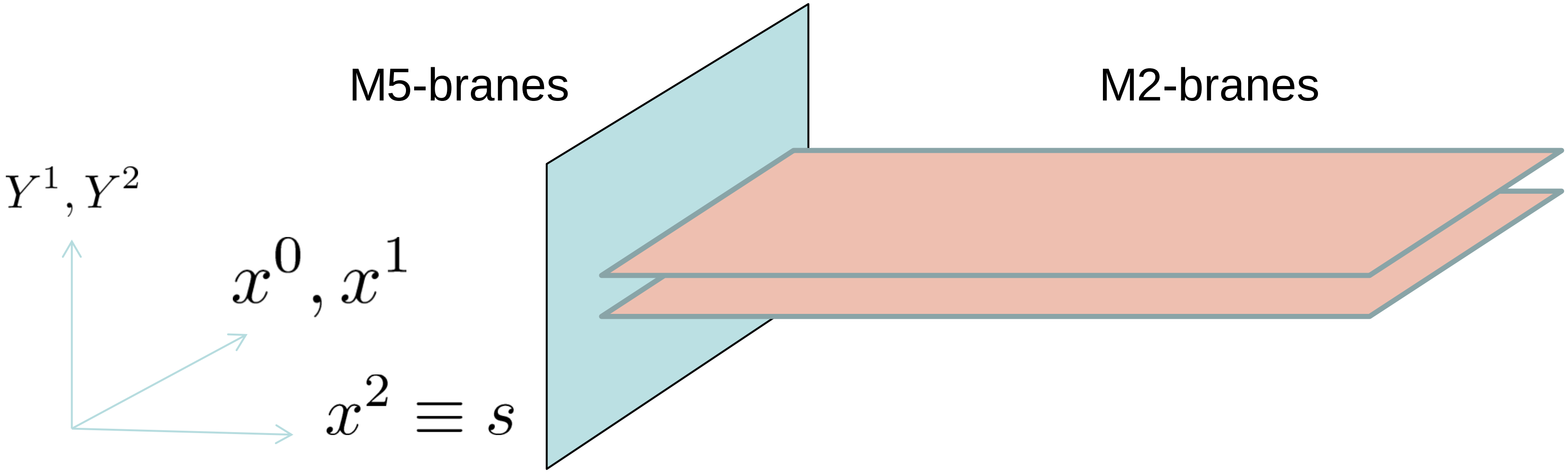
½ BPS equations for M2-M5 bound stare is given by

$$\dot{Y}^a = Y^b Y^{b\dagger} Y^a - Y^a Y^{b\dagger} Y^b$$

where $\dot{Y} \equiv \frac{dY}{ds}$ $(a = 1, 2)$

ST

Gomis-Rodriguez-Gomez-Van Raamsdonk-Verlinde



If $Y^a(c) = \infty$, there will be M5-brane at $x^2 = c$

If $Y^a(c) = \infty$, there will be M5-brane at $x^2 = c$

Actually, any solution will becomes
the following basic solution near the M5-branes:

$$Y^a = \sqrt{\frac{k}{4\pi x^2}} S^a$$

where S are constant N x N matrices satisfying

$$\begin{aligned} S^1 &= S^2 S^{2\dagger} S^1 - S^1 S^{2\dagger} S^2 \\ S^2 &= S^1 S^{1\dagger} S^2 - S^2 S^{1\dagger} S^1 \end{aligned}$$

This can be solved by digonalizing S^1
by U(N) x U(N) gauge symmetry

$$(S^1)_{ij} = \delta_{i,\,j-1}\sqrt{i}, \;\; (S^2)_{ij} = \delta_{ij}\sqrt{N-i} \;\; (i,j = 1,\cdots,N)$$

$$\begin{pmatrix} 0 & \sqrt{N-1} & & & \\ & 0 & \ddots & & \\ & & \ddots & \sqrt{2} & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \sqrt{N-2} & \\ & & & & \sqrt{N-1} \end{pmatrix}$$

This should represent a Fuzzy 3-sphere

Because $\begin{cases} Y^a = \sqrt{\dfrac{k}{4\pi x^2}} S^a \\ \dfrac{1}{N} Tr\left((S^a)^\dagger S_a \right) = N - 1 \end{cases}$

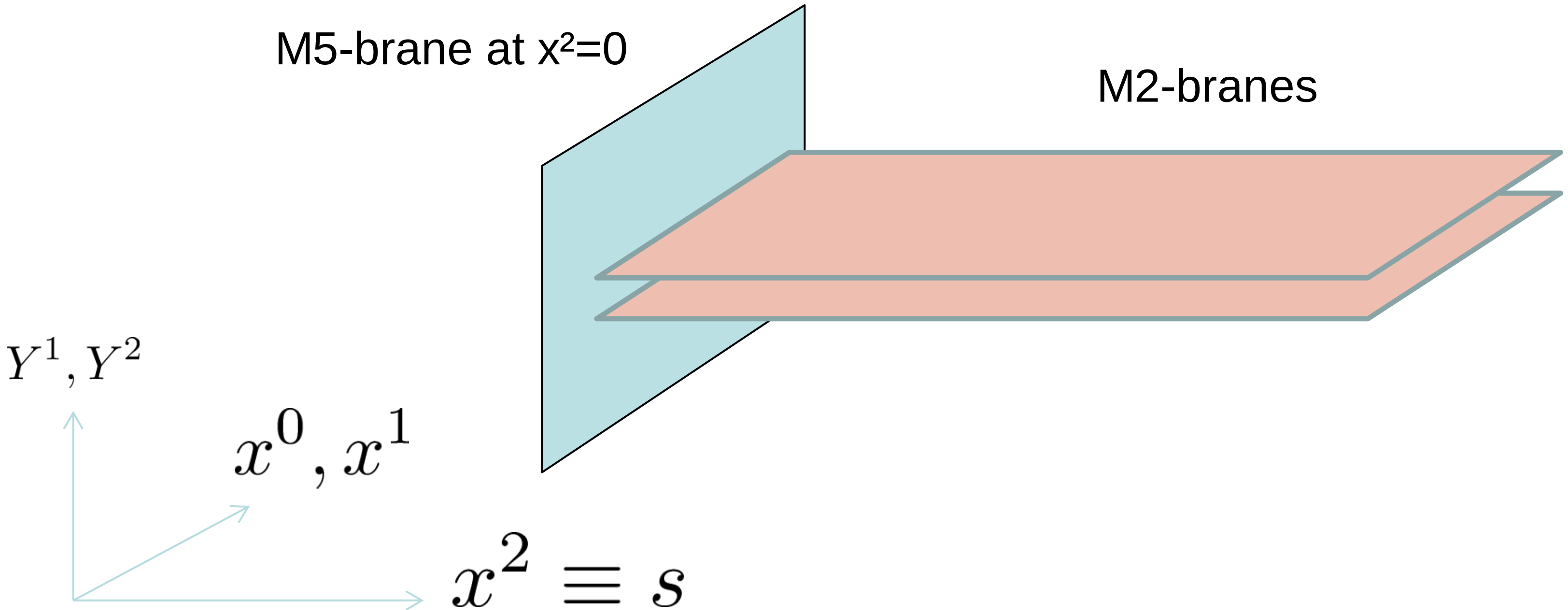
radius of 3-sphere is $r \sim \sqrt{kN/(4\pi x^2)}$

The action is evaluated as

$$S \sim -2 \int d^3x \mathrm{Tr} D_\mu Y_a^\dagger D^\mu Y^a \sim -2 \int d^3x \frac{k}{16\pi (x^2)^3} \mathrm{Tr}(S^a (S^a)^\dagger)$$
$$\sim - \int dx^0 dx^1 \boxed{dr r^3} \frac{2\pi}{k}$$

Correct tension of M5-brane!

The basic solution $Y^a = \sqrt{\frac{k}{4\pi x^2}} S^a$



Adding to this, there are some few solutions had been known

Translational symmetry

If T is a solution of the Nahm equation $\dot{T}^I = i\epsilon_{IJK} T^J T^K$

then, $T^I + \mathrm{const}.1$ is also a solution.

But, even if Y is a solution of the BPS equation

$$\dot{Y}^a = Y^b Y^{b\dagger} Y^a - Y^a Y^{b\dagger} Y^b$$

$Y^a + \mathrm{const}.1$ is NOT necessary a solution.

Actually, translational symmetry is broken by the orbifolding.
This is an origin of additional difficulty to solve the equation.

The Lax pair

Sakai-ST

Let us consider the BPS equations:

$$\dot{Y}^a = Y^b Y^{b\dagger} Y^a - Y^a Y^{b\dagger} Y^b$$

$$(a = 1, 2)$$

The symmetry of the BPS equations is

$$Y^a \to Y'^a = e^{i\varphi} \Lambda^a{}_b U Y^b V^\dagger$$

$$U, V \in \mathrm{SU}(N), \quad (\Lambda^a{}_b) \in \mathrm{SU}(2), \quad e^{i\varphi} \in \mathrm{U}(1)$$

$$\dot{Y}^a = Y^b Y^{b\dagger} Y^a - Y^a Y^{b\dagger} Y^b$$

equivalent!

The Lax equation $\dot{A} = [A, B]$

$$A(s;\lambda) = \begin{pmatrix} O & Y^1 + \lambda Y^2 \\ Y^{1\dagger} - \lambda^{-1} Y^{2\dagger} & O \end{pmatrix},$$

$$B(s;\lambda) = \begin{pmatrix} \lambda^{-1} Y^1 Y^{2\dagger} + \lambda Y^2 Y^{1\dagger} & O \\ O & \lambda Y^{1\dagger} Y^2 + \lambda^{-1} Y^{2\dagger} Y^1 \end{pmatrix}$$

λ is a arbitrary constant parameter

Because of $\dot{A} = [A, B]$

$\mathrm{Tr} A^m$ are "conserved charge"

These are summarized to
the spectral curve $P(\mu, \lambda) = 0$
defined by

$$
\begin{aligned}
P &:= \det(\eta \mathbf{1}_{2N} - A) \\
&= \det\left[\eta^2 \mathbf{1}_N - \left(Y^1 + \lambda Y^2\right)\left(Y^{1\dagger} - \lambda^{-1} Y^{2\dagger}\right)\right] \\
&= \det\left[\eta^2 \mathbf{1}_N - \left(Y^{1\dagger} - \lambda^{-1} Y^{2\dagger}\right)\left(Y^1 + \lambda Y^2\right)\right]
\end{aligned}
$$

$$\mu := \eta^2$$

We introduce a “chirality” matrix

$$\Gamma := \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{pmatrix}$$

Then, we find

$$\{A, \Gamma\} = 0, \qquad [B, \Gamma] = 0$$

$$\begin{cases} A(s;\lambda) = \begin{pmatrix} O & Y^1 + \lambda Y^2 \\ Y^{1\dagger} - \lambda^{-1} Y^{2\dagger} & O \end{pmatrix} \\ B = \lambda \dfrac{\partial}{\partial \lambda} A^2 \end{cases}$$

We also introduce a star-conjugate:

$$\mathcal{M}^\star(\lambda) := \mathcal{M}(-\bar{\lambda}^{-1})^\dagger$$

Then, we find

$$A^\star = A, \qquad B^\star = -B$$

$$\begin{cases} A(s;\lambda) = \begin{pmatrix} O & Y^1 + \lambda Y^2 \\ Y^{1\dagger} - \lambda^{-1} Y^{2\dagger} & O \end{pmatrix} \\ B = \lambda \dfrac{\partial}{\partial \lambda} A^2 \end{cases}$$

Now, we will consider the so-called
linear problem:

$$A(s;\lambda)\psi(s;\lambda) = \eta(\lambda)\psi(s;\lambda),$$
$$B(s;\lambda)\psi(s;\lambda) = -\dot{\psi}(s;\lambda).$$

If ψ is an eigen vector with eigen value η

then $\Gamma\psi$ is an eigen vector with eigen value $-\eta$

Then, we will take

$$\psi_{N+m} = \Gamma\psi_m, \qquad \eta_{N+m} = -\eta_m, \qquad m = 1, \ldots, N$$

We define an N x 2N matrix and an 2N x 2N matrix:

$$\Psi := (\psi_1, \ldots, \psi_{2N}) = (\psi_1, \ldots, \psi_N, \Gamma\psi_1, \ldots, \Gamma\psi_N),$$

$$D := \mathrm{diag}(\eta_1, \ldots, \eta_{2N}) = \mathrm{diag}(\eta_1, \ldots, \eta_N, -\eta_1, \ldots, -\eta_N)$$

Then, the linear problems are written as:

$$A\Psi = \Psi D \qquad B\Psi = -\dot{\Psi}$$

We have assumed there are 2N linearly independent solutions for $A\psi = \eta\psi$

Then, we can reconstruct A(s;λ) from Ψ(s;λ) and D. Indeed, we find

$$A(s;\lambda) = \Psi(s;\lambda) C(\lambda) \Psi^{\star}(s;\lambda)$$

$$C := D\mathcal{N}^{-1}$$

$$\mathcal{N} := \Psi^{\star}\Psi$$

The relation to two Nahm equations

Each of these 3 matrices

$$T_1^I := (\sigma^I)_{ab} Y^a Y^{b\dagger}$$

$$T_2^I := (\sigma^I)_{ab} Y^{b\dagger} Y^a$$

satisfy the Nahm equation $\dot{T}^I = i\epsilon_{IJK} T^J T^K$

Nosaka-ST

the Nahm equation $\dot{T}^I = i\epsilon_{IJK} T^J T^K$

also has a Lax representation

$$\dot{A}_\alpha = [A_\alpha, B_\alpha]$$

$$A_\alpha := T^3_\alpha + \frac{\lambda}{2}\left(T^1_\alpha - iT^2_\alpha\right) - \frac{1}{2\lambda}\left(T^1_\alpha + iT^2_\alpha\right)$$
$$B_\alpha := \frac{\lambda}{2}\left(T^1_\alpha - iT^2_\alpha\right) + \frac{1}{2\lambda}\left(T^1_\alpha + iT^2_\alpha\right).$$

relation to Lax pair for the BPS equations in ABJM?

Indeed, a simple relation:

$$A^2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

similar to "Dirac equation"!

These relations means

$$P = \det(\mu \mathbf{1}_N - A_1) = \det(\mu \mathbf{1}_N - A_2)$$

The linear problems for the Nahm equations:

$$A_\alpha \Psi_\alpha = \Psi_\alpha M$$

$$B_\alpha \Psi_\alpha = -\dot{\Psi}_\alpha$$

$$M = \mathrm{diag}(\mu_1, \ldots, \mu_N)$$

Now consider the original BPS eq. in ABJM and the linear problem $A\Psi = \Psi D$

If we express $$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \Psi_1 & \Psi_1 \\ \Psi_2 & -\Psi_2 \end{pmatrix}$$

Ψ_1, Ψ_2 are the eigenvectors of A for the corresponding Nahm equations

$$D = \begin{pmatrix} H & O \\ O & -H \end{pmatrix}, \qquad H = \mathrm{diag}(\eta_1, \ldots, \eta_N)$$

$$H^2 = M$$

Next, we assume eigenvectors for Nahm data are given:

$$\Psi_1, \Psi_2, M$$

⇩

$$D = \begin{pmatrix} H & O \\ O & -H \end{pmatrix}, \qquad H = \mathrm{diag}(\eta_1, \ldots, \eta_N)$$

$$H^2 = M$$

A candidate for Ψ should be

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_1 & \Psi_1 \\ \Psi_2 & -\Psi_2 \end{pmatrix}$$

For this Ψ, we can reconstruct A as

$$A = \begin{pmatrix} O & \Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star \\ \Psi_2 H \mathcal{N}_1^{-1} \Psi_1^\star & O \end{pmatrix}$$

Comparing with $A(s;\lambda) = \begin{pmatrix} O & Y^1 + \lambda Y^2 \\ Y^{1\dagger} - \lambda^{-1} Y^{2\dagger} & O \end{pmatrix}$

$$\text{Cond.1} \quad \frac{\partial^2}{\partial \lambda^2} \left[\Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star \right] = 0$$

$$\text{Cond.2} \quad H \mathcal{N}_1 = \mathcal{N}_2 H \quad \Longleftarrow \quad A^\star = A$$

With these conditions, this A gives the solutions of the BPS equations!

The solutions for two M2-branes

Sakai-ST

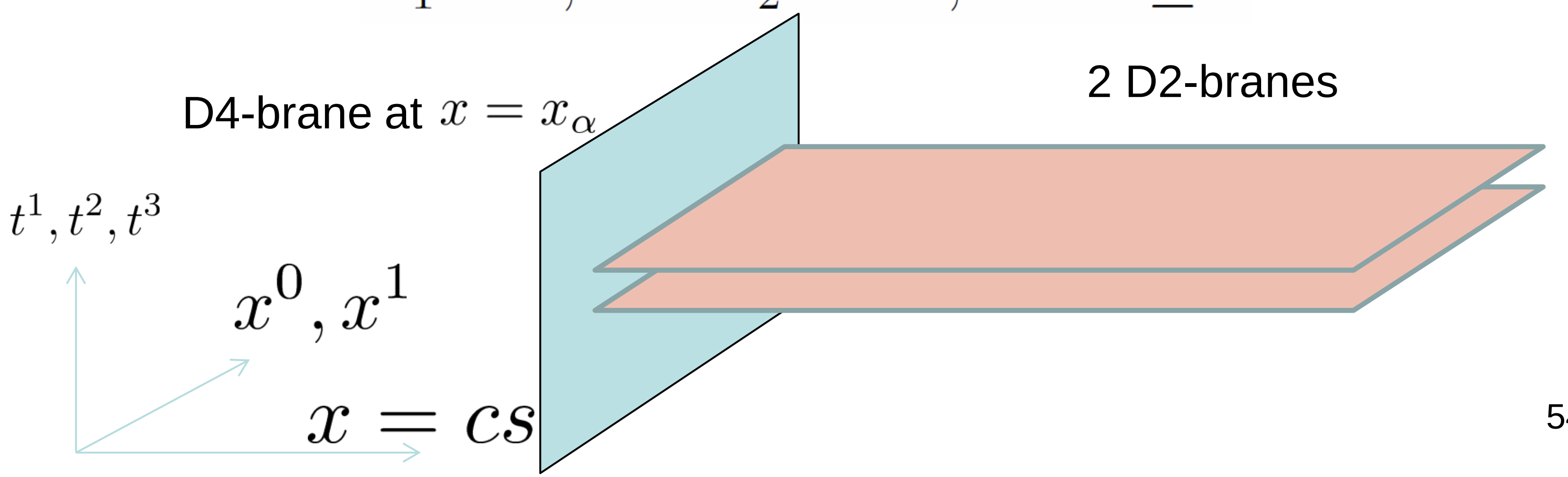
General solutions of the Nahm equations for N=2 case (with a D4-brane)

$$T^1_\alpha = \frac{c}{\sinh(x - x_\alpha)}\frac{\sigma^1}{2} + t^1\mathbf{1}_2, \qquad T^2_\alpha = \frac{c}{\sinh(x - x_\alpha)}\frac{\sigma^2}{2} + t^2\mathbf{1}_2$$

$$T^3_\alpha = \frac{c}{\tanh(x - x_\alpha)}\frac{\sigma^3}{2} + t^3\mathbf{1}_2.$$

$$x = cs, \qquad c \geq 0$$

$$x_1 = 0, \qquad x_2 = -l, \qquad l \geq 0$$



For this, the Lax pair is written as

$$A_\alpha = \left(\tanh\frac{x - x_\alpha}{2}\right)^{-\rho^1} M \left(\tanh\frac{x - x_\alpha}{2}\right)^{\rho^1}$$

$$\begin{cases} M = \left(\frac{c}{2}\sigma^3 + t_\lambda \mathbf{1}_2\right) \\ \rho^1 = \frac{\lambda + \lambda^{-1}}{4}\sigma^1 + \frac{\lambda - \lambda^{-1}}{4i}\sigma^2 = \frac{1}{2}\begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix} \\ t_\lambda = t^3 + \frac{\lambda}{2}(t^1 - it^2) - \frac{1}{2\lambda}(t^1 + it^2) \end{cases}$$

$$\Longrightarrow \quad \Psi_\alpha = \left(\tanh\frac{x - x_\alpha}{2}\right)^{-\rho^1} D_\alpha$$

$$D_\alpha^\star D_\alpha = \mathbf{1}_2$$

We can compute the candidate of A as

$$\Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star = \left(\tanh\frac{x-x_1}{2}\right)^{-\rho^1} D_1 M^{1/2} D_2^\star \left(\tanh\frac{x-x_2}{2}\right)^{\rho^1}$$

$$\left(\tanh\frac{x}{2}\right)^{\pm\rho^1} = \frac{1}{\sqrt{2\sinh x}}\begin{pmatrix} e^{x/2} & \mp\lambda^{-1}e^{-x/2} \\ \mp\lambda e^{-x/2} & e^{x/2} \end{pmatrix}$$

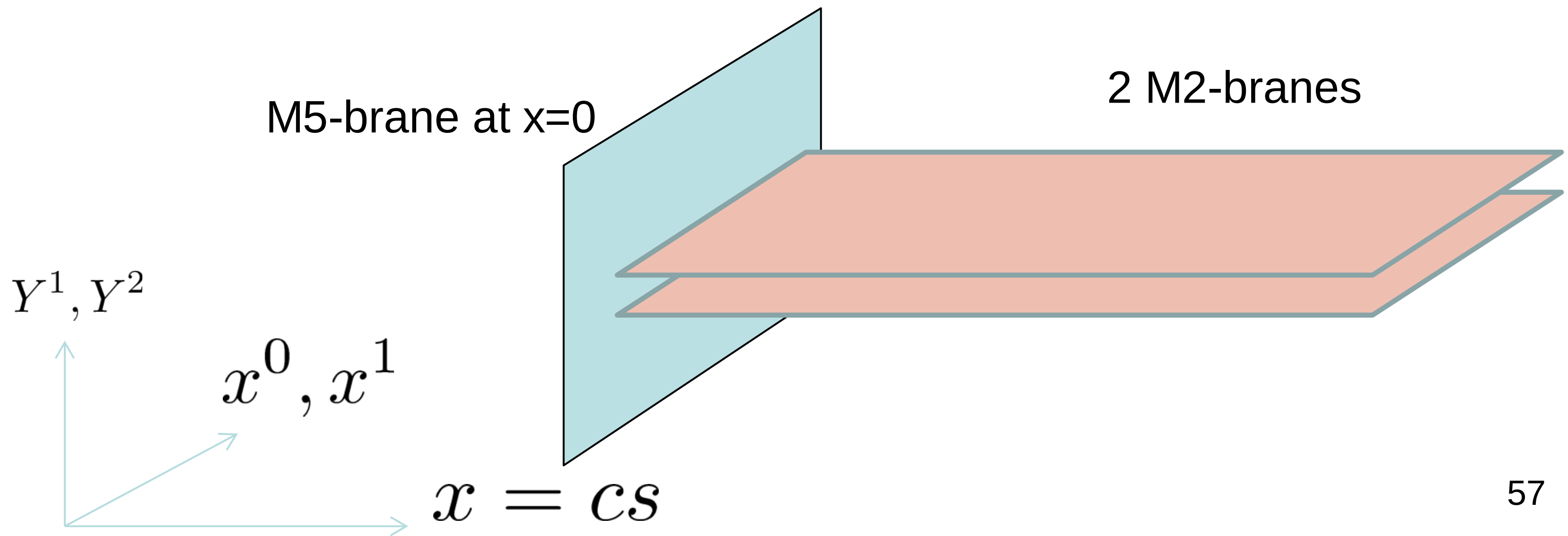
Then, we require the conditions 1,2

$$\frac{\partial^2}{\partial\lambda^2}\left[\Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star\right] = 0$$

$$A^\star = A$$

Finally, we find general solution for N=2

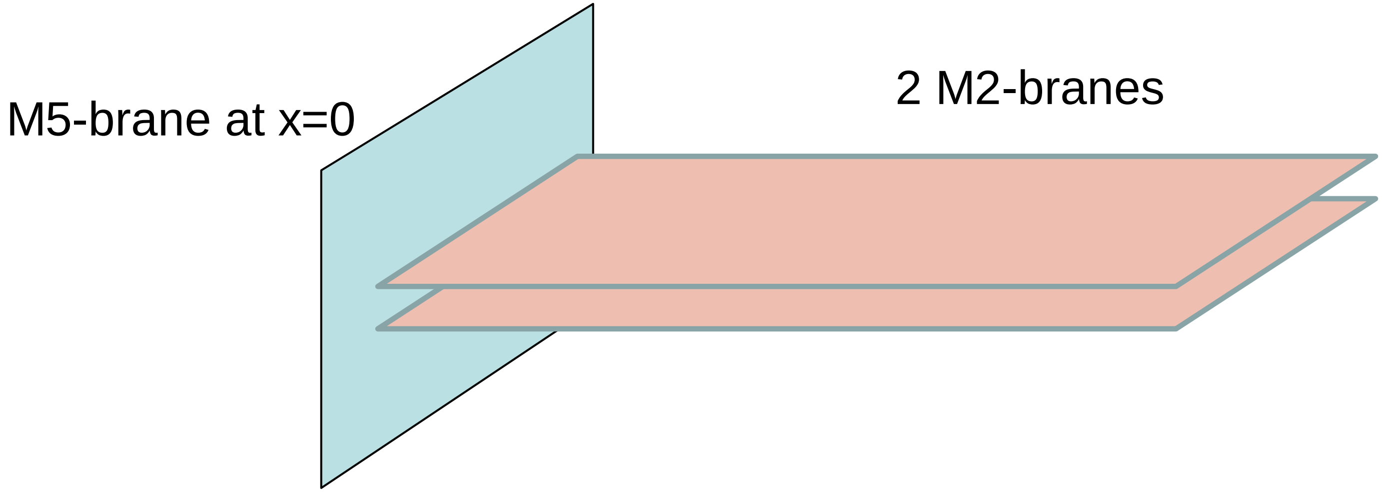
$$\begin{cases} Y^1 = \sqrt{\dfrac{c}{2\sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh(x+l)\cos\frac{\theta}{2}e^{i\phi} & \sinh l \sin\frac{\theta}{2} \\ 0 & \sinh x \cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \\ Y^2 = \sqrt{\dfrac{c}{2\sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh x \sin\frac{\theta}{2} & 0 \\ \sinh l \cos\frac{\theta}{2}e^{i\phi} & \sinh(x+l)\sin\frac{\theta}{2} \end{pmatrix} \end{cases}$$



$$\begin{cases} Y^1 = \sqrt{\dfrac{c}{2\sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh(x+l)\cos\frac{\theta}{2}e^{i\phi} & \sinh l \sin\frac{\theta}{2} \\ 0 & \sinh x \cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \\ Y^2 = \sqrt{\dfrac{c}{2\sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh x \sin\frac{\theta}{2} & 0 \\ \sinh l \cos\frac{\theta}{2}e^{i\phi} & \sinh(x+l)\sin\frac{\theta}{2} \end{pmatrix} \end{cases}$$

$$x \to 0 \qquad Y^a \to \text{basic solution}$$

$$x \to \infty \quad \begin{cases} Y^1 \to \sqrt{\dfrac{c}{2\sinh l}} \cos\dfrac{\theta}{2} e^{i\phi}\ \mathrm{diag}(e^{l/2}, 1) \\ Y^2 \to \sqrt{\dfrac{c}{2\sinh l}} \sin\dfrac{\theta}{2}\ \mathrm{diag}(1, e^{l/2}) \end{cases}$$



We can rewrite the solution as

$$Y^1 = \frac{1}{2}\left(f_1 \sin\frac{\theta}{2}\,\sigma^1 + f_2 \sin\frac{\theta}{2}\,i\sigma^2 + f_3 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^3 - f_0 e^{i\phi}\cos\frac{\theta}{2}\,\mathbf{1}_2\right)$$
$$Y^2 = \frac{1}{2}\left(f_1 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^1 - f_2 e^{i\phi}\cos\frac{\theta}{2}\,i\sigma^2 - f_3 \sin\frac{\theta}{2}\,\sigma^3 - f_0 \sin\frac{\theta}{2}\,\mathbf{1}_2\right)$$

$$f_1 = f_2 = \sqrt{\frac{c\sinh l}{2\sinh x \sinh(x+l)}}, \quad f_3 = \frac{\cosh(x+l/2)}{\cosh(l/2)} f_1, \quad f_0 = -\frac{\sinh(x+l/2)}{\sinh(l/2)} f_1.$$

We can show that

$$Y^1 = \frac{1}{2}\left(f_1 \sin\frac{\theta}{2}\,\sigma^1 + f_2 \sin\frac{\theta}{2}\,i\sigma^2 + f_3 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^3 - f_0 e^{i\phi}\cos\frac{\theta}{2}\,\mathbf{1}_2\right)$$

$$Y^2 = \frac{1}{2}\left(f_1 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^1 - f_2 e^{i\phi}\cos\frac{\theta}{2}\,i\sigma^2 - f_3 \sin\frac{\theta}{2}\,\sigma^3 - f_0 \sin\frac{\theta}{2}\,\mathbf{1}_2\right)$$

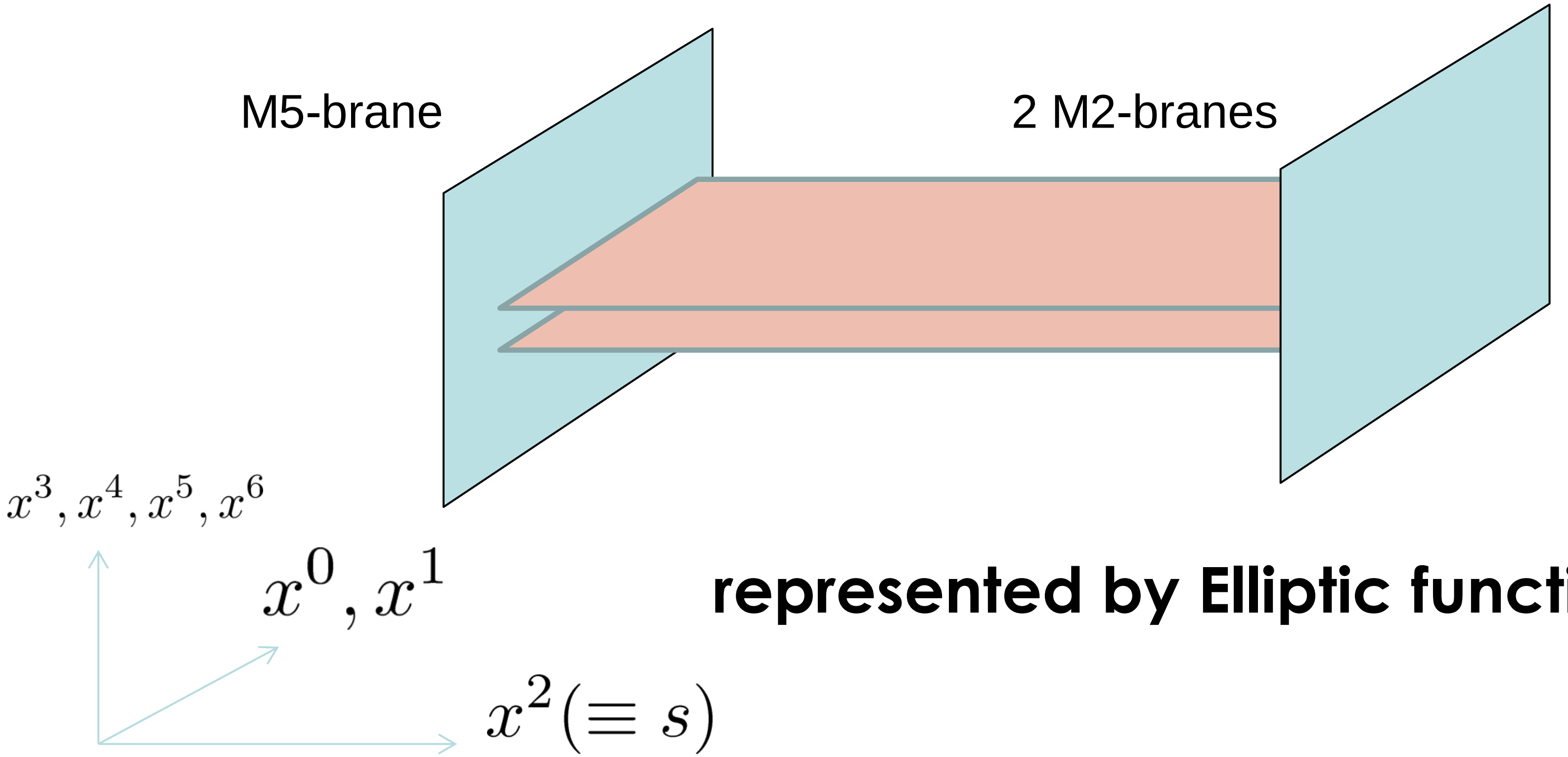
with

$$\dot{f}_i = f_j f_k f_l$$

are the solutions of the BPS equations.

$$\Longrightarrow \quad f_I^2 - f_0^2 \text{ are constants}$$

These include
all solutions for two M2-branes:



represented by Elliptic functions

explicit form 1

$$f_i = \frac{\vartheta_{i+1}(u)}{\vartheta_{i+1}(u_*)} \sqrt{\frac{\pi}{2\omega_1} \frac{\vartheta_1(u_*)\vartheta_2(u_*)\vartheta_3(u_*)\vartheta_4(u_*)}{\vartheta_1(u_* + u)\vartheta_1(u_* - u)}}$$

where

$\vartheta_i(u) := \vartheta_i(u, \tau)$ are Jacobi theta functions

$$u = \frac{s - s_0}{2\omega_1} \qquad -u_* < u < u_*$$

$$s_0 \in \mathbb{R}, \qquad 0 < u_* < \frac{1}{2}, \qquad \omega_1 \in \mathbb{R}_{>0}, \qquad \tau \in i\mathbb{R}_{>0}$$

explicit form 2

$$f_0 = \left(\frac{\wp_1(s_*)\wp_2(s_*)\wp_3(s_*)}{\wp(s-s_0)-\wp(s_*)} \right)^{1/2}, \qquad f_I = \frac{\wp_I(s-s_0)}{\wp_I(s_*)} f_0 \quad (I=1,2,3)$$

where

$$\left\{ \begin{aligned} & s_* = 2\omega_1 u_*, \qquad 0 < s_* < \omega_1 \\ & f_I^2 - f_0^2 = \frac{\pi \vartheta_{I+1}^2}{2\omega_1} \frac{\vartheta_{J+1}(u_*)\vartheta_{K+1}(u_*)}{\vartheta_1(u_*)\vartheta_{I+1}(u_*)} \\ & \qquad = \frac{\wp_J(s_*)\wp_K(s_*)}{\wp_I(s_*)} \\ & \qquad =: a_I^2 \qquad (a_I > 0), \end{aligned} \right.$$

explicit form 3

$$f_0 = \frac{a_3 \operatorname{sn} x}{\sqrt{\operatorname{sn}^2 x_* - \operatorname{sn}^2 x}}, \qquad f_1 = \frac{a_1 \operatorname{sn} x_* \operatorname{cn} x}{\sqrt{\operatorname{sn}^2 x_* - \operatorname{sn}^2 x}}$$

$$f_2 = \frac{a_2 \operatorname{sn} x_* \operatorname{dn} x}{\sqrt{\operatorname{sn}^2 x_* - \operatorname{sn}^2 x}}, \qquad f_3 = \frac{a_3 \operatorname{sn} x_*}{\sqrt{\operatorname{sn}^2 x_* - \operatorname{sn}^2 x}}$$

where $x = c(s - s_0), \qquad c = a_2\sqrt{a_1^2 - a_3^2}, \qquad \operatorname{sn} x_* = \sqrt{1 - \frac{a_3^2}{a_1^2}}$

Corresponding Nahm data:

$$T_\alpha^I = \wp_I\left(s - s_\alpha\right)\frac{\sigma^I}{2} + \frac{n_I}{4}\left(a_I^2 - a_J^2 - a_K^2\right)\mathbf{1}_2$$

where

$$\wp_1(\tilde{s}) = c\frac{\operatorname{cn} x}{\operatorname{sn} x}, \qquad \wp_2(\tilde{s}) = c\frac{\operatorname{dn} x}{\operatorname{sn} x}, \qquad \wp_3(\tilde{s}) = c\frac{1}{\operatorname{sn} x}$$

$$s_1 = s_0 - s_*, \qquad s_2 = s_0 + s_*$$

Conclusion

- **BPS M5-branes in ABJM**
- **Lax representation**
- **Dirac equation like structure**
- **New solutions**
- **All solutions for 2 M2-branes**

Future works

- **Other M2-brane action**
- **Other solutions using integrability**
- **Nahm transformation to solutions in mysterious M5-brane action**
- **3-algebra structure**
- **Moduli space metric, etc**

Fin.