Instantons defined by Lie groups

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Old references

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4-dimensional origins

On an oriented Riemannian 4-manifold,

$$\Lambda^2 T^* M = \Lambda^2_+ \oplus \Lambda^2_-$$

since $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$, and there is an elliptic complex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d_-} \Omega^2_- \to 0.$$

If Q is a principal $\mathrm{SU}(2)$-bundle with self-dual connection A with (so $F = dA + A \wedge A \in \Omega^2_+$ and $*F = F$) then H^1 of the complex

$$0 \to \Omega^0(\mathrm{ad}\, Q) \longrightarrow \Omega^1(\mathrm{ad}\, Q) \longrightarrow \Omega^2_-(\mathrm{ad}\, Q) \to 0$$

captures infinitesimal deformations modulo gauge equivalence.

The index is $8k - 3$, and a framed moduli space of dimension $8k$ is hyperkähler (meaning holonomy $Sp(k)$), given by HK quotients $\mathbb{H}^{k(k+3)/2} /\!/\!/ O(k) \cong \mu_\infty^{-1}(0)/\mathscr{G}$ [D].

Riemannian G-structures

More generally, for $G \subset \mathrm{SO}(n)$, we can write

$$\Lambda^2 \;\cong\; \mathfrak{so}(n) \;=\; \mathfrak{g} \oplus \mathfrak{g}^\perp.$$

Given a G-structure on M (in fact, an $N(G)$-structure) this decomposition passes to 2-forms.

Definition. In this context, an *instanton* is a connection (on a bundle over M) whose curvature 2-forms F_i^j lie in $\mathfrak{g}$.

Example. If the holonomy reduces to G then the Levi-Civita connection is an instanton because R_{ijkl} belongs to the kernel of

$$S^2(\mathfrak{g}) \;\subset\; \mathfrak{g} \otimes \Lambda^2 \;\subset\; \Lambda^2 \otimes \Lambda^2 \;\longrightarrow\; \Lambda^4.$$

On the other hand, the Killing form in $S^2(\mathfrak{g})$ maps to a *non-zero* 4-form unless it represents curvature of a Riemannian symmetric space. So 4-forms arise in abundance!

Hitchin-Kobayashi correspondence

If $G = \mathrm{SU}(n) \subset \mathrm{SO}(2n)$ so $N(G) = \mathrm{U}(n)$ then

$$\Lambda^2 = [\![\Lambda^{2,0}]\!] \oplus [\Lambda_0^{1,1}] \oplus \langle\omega\rangle\,,$$

and $\mathfrak{g} = \mathfrak{su}(n) \cong [\Lambda_0^{1,1}]$. An instanton is a connection with $(1,1)$ curvature and vanishing trace $F \wedge \omega^{n-1}$, though this would force $c_1 = 0$. More generally we require that the trace be a (constant) multiple of the identity, the Hermitian-Einstein condition.

Over a complex manifold:

- A connection with $(1,1)$ curvature on a vector bundle renders it a *holomorphic* bundle [cf. NN].
- A holomorphic bundle admits a unique connection with $(1,1)$ curvature compatible with a given *Hermitian metric* on its fibres.

Theorem [D,UY]. On a compact Kähler manifold, an irreducible holomorphic vector bundle admits a HE connection iff it is *stable*.

Using a 3-form

If $G = \mathrm{G}_2 \subset \mathrm{SO}(7)$ so $N(G) = G$ then

$$\begin{array}{rcll} \Lambda^2 & = & \Lambda^2_{14} \oplus \Lambda^2_7 & \cong \ \mathfrak{g}_2 \oplus \Lambda^1 \\ \Lambda^3 & = & \Lambda^3_{27} \oplus \Lambda^3_7 \oplus \langle \varphi \rangle & \cong \ S^2_0 \oplus \Lambda^1 \oplus \mathbb{R}. \end{array}$$

Example. If $\varphi = (12-34)5 + (13-42)6 + (14-23)7 + 567$, we have $12{+}34,\ 13{+}24,\ 14{+}23 \in \Lambda^2_+ \subset \mathfrak{g}_2$.

An instanton is characterized by the equivalent equations

$$F_7 = 0, \qquad F \wedge (*\varphi) = 0, \qquad F \wedge \varphi = *F.$$

Instantons are YM connections because

$$c_2(F) \cup [\varphi] = \int F \wedge F \wedge \varphi = 4\|F_{14}\|^2 - 18\|F_7\|^2$$

and $\|F\|^2$ has an absolute minimum if $F_7 = 0$.

grad, curl, div in dim 7

Suppose that M^7 has a G_2 structure. Consider

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{D_1} \Omega^2_7 \xrightarrow{D_2} \Omega^3_1 \to 0$$

where $D_1 = \pi_7 \circ d$ arises from the cross product. It is complex iff

$$D_2 \circ D_1 = 0 \iff d(\Omega^2_{14}) \subseteq \langle \varphi \rangle^\perp \Longleftarrow d * \varphi = 0.$$

Lemma [CN]. If M^7 is oriented and spin it has a G_2 structure, indeed one with $d * \varphi = 0$.

Given an instanton on a bundle Q, we can extend the operators so $D \circ D = F_7 = 0$ and obtain an elliptic complex

$$0 \to \Omega^0(\mathrm{ad}\, Q) \to \Omega^1(\mathrm{ad}\, Q) \to \Omega^1(\mathrm{ad}\, Q) \to \Omega^0(\mathrm{ad}\, Q) \to 0$$

whose H^1 parametrizes infinitesimal deformations. Close analogue with the de Rham complex $1 \to 3 \to 3 \to 1$ over a 3-manifold.

Integrability

We shall construct a differential complex for any $G \subset \mathrm{SO}(n)$ that begins $0 \to \Omega^0 \to \Omega^1 \to \cdots$

Set $\Lambda^{-2} = \Lambda^{-1} = 0$ and $A^k = (\mathfrak{g} \wedge \Lambda^{k-2})^\perp$. Define

$$D\colon \quad \mathscr{A}^k \subset \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{\pi} \mathscr{A}^{k+1}.$$

Here $\mathscr{A}^k = \Gamma(M, A^k)$, so $\mathscr{A}^k = \Omega^k$ for $k = 0, 1$.

Proposition. $D^2 = 0$ if only only if $d\colon \Omega^2 \to \Omega^3$ maps sections of $\mathfrak{g}$ to those of $\mathfrak{g} \wedge \Lambda^1$.

This is obviously true if the holonomy of M lies in $N(G)$; in general it is a condition on the *intrinsic torsion* $\tau \in \Gamma(M, \mathfrak{g} \otimes \Lambda^1)$.

In any case, we would like

$$0 \to \mathscr{A}^0 \to \mathscr{A}^1 \to \mathscr{A}^2 \to \cdots \to 0$$

to be *elliptic*. It is when G equals $\mathrm{SU}(n), \mathrm{G}_2, \mathrm{Spin}\,7, \mathrm{Sp}(n), \ldots$

Nearly-Kähler 6-manifolds

If $G = \mathrm{SU}(3)$ so $N(G) = \mathrm{U}(3)$, the complex becomes

$$0 \to \Omega^0 \to \Omega^1 \to \Gamma([\![\Lambda^{2,0}]\!] \oplus \langle\omega\rangle) \to \Gamma([\![\Lambda^{3,0}]\!]) \to 0.$$

with dimensions $1 \to 6 \to 7 \to 2$, provided *most* of the Nijenhuis tensor vanishes! We get a theory of instantons over nearly-Kähler 6-manifolds (meaning $(\nabla_X J)X = 0$)).

Example. The twistor spaces $\mathbb{CP}^3 \to S^4$ and $\mathbb{F}^3 \to \mathbb{CP}^2$ both have a $\mathrm{U}(1)$-connection A_1 whose curvature F_1 is a Kähler form and another NK 2-form F_2 such that $F_2 \wedge F_2 = d\psi$ with $F_1 \wedge \psi = 0$.

$$\implies 0 = d(F_1 \wedge \psi) = dF_1 \wedge \psi + F_1 \wedge (F_2 \wedge F_2) = 2F_1 \wedge (*F_2).$$

Thus A_1 is an instanton for the NK metric.

Quaternionic case

If $G = \mathrm{Sp}(n) \subset \mathrm{SO}(4n)$ so $N(G) = \mathrm{Sp}(n)\mathrm{Sp}(1)$ and

$$\begin{aligned} (\Lambda^1)_c &= E \otimes H \qquad (\text{cf. } S \otimes \widetilde{S}) \\ (\Lambda^2)_c &= S^2E \oplus S^2H \oplus (\Lambda_0^2 E \otimes S^2 H) \\ &\cong \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \oplus \mathfrak{m}. \end{aligned}$$

Manifolds M^{4n} $(n \geqslant 2)$ like $\mathbb{HP}^n$ with holonomy in $N(G)$ are quaternion-Kähler and behave as if they were *nearly hyperkähler*.

Since $S^2E = \Lambda_I^{1,1} \cap \Lambda_J^{1,1} \oplus \Lambda_K^{1,1}$, instantons give rise to holomorphic bundles over the twistor space Z^{2n+1}, which fibres over M.

Using a 4-form again, one shows that the Yang-Mills functional has a critical point whenever any 2 of the 3 components vanish.

Not an abs max/min if $F \in \Gamma(M, \mathfrak{m})$, but no examples known.

Instantons via quaternions

Take $M^{4n} = \mathbb{HP}^n$. Let $\mathbf{q} = (q_0, q_1, \ldots, q_n) \in \mathbb{H}^{n+1} \setminus 0$, $m = [\mathbf{q}]$.

A linear form $\sum a_r q_r$ (with $a_r \in \mathbb{H}$) defines a section of the tautological line bundle H (fibre $\mathbb{H} = \mathbb{C}^2$) inside $\underline{\mathbb{H}}^{n+1}$, and

$$E_m = \ker\left(\mathbf{q}^\top : \mathbb{H}^{n+1} \to H_m\right), \qquad \text{so} \quad E = H^\perp.$$

Take matrices $A_0, A_1, \ldots, A_n \in \mathbb{H}^{n+k,k}$ and set $\mathbb{A} = \sum_{r=0}^n A_r q_r$.

Theorem If $A_r^* A_s$ is symmetric for all r, s and $\mathbb{A}$ has rank k for all $\mathbf{q} \neq 0$ then $\ker \mathbb{A}$ is an $\mathrm{Sp}(n)$ instanton on $\mathbb{HP}^n$.

Proof. Relies on the fact that the real components of

$$dq_r \wedge d\overline{q}_s = (du_r + j\, dv_r) \wedge (d\overline{u}_r - d\overline{v}_r\, j)$$

span the subspace $\mathfrak{sp}(n)$ of Λ^2. Projection to $\ker \mathbb{A}$ equals $\Pi = 1 - \mathbb{A}(\mathbb{A}^*\mathbb{A})^{-1}\mathbb{A}^*$ and the induced curvature is $\Pi(d\Pi \wedge d\Pi)\Pi$.

The twistor space

. . . of $\mathbb{HP}^n$ is $\mathbb{CP}^{2n+1}$, which is the total space of $\begin{array}{c}\mathbb{P}_c(H)\\ \downarrow\\ \mathbb{HP}^n.\end{array}$

The instantons $F = \ker \mathbb{A}$ pull back to holomorphic bundles $\pi^* F$ (fibre $\mathbb{C}^{2n}$) with $c(F) = (1-h)^{-k}$, characterized by

$$H^q(\mathbb{CP}^{2n+1}, \pi^* F \otimes \mathscr{O}(p)) = 0 \quad \begin{cases} q = 1, & p \leqslant -2 \\ 2 \leqslant q \leqslant n, & p \in \mathbb{Z}. \end{cases}$$

Example. For $n = k = 2$, we can take $\mathbb{A} = \begin{pmatrix} q_0 & 0 \\ 0 & q_0 + q_2 \\ q_1 & q_2 \\ q_2 & q_1 \end{pmatrix}$.

The deformation complex for $n = 1$ has $h^1 = 8k - 3$.

Proposition. If $n = 2$ the deformation complex of the instantons above has $h^1 - h^2 = \frac{3}{2}k(17-k) - 10 = 14, 35, 53, \ldots$

4-forms in 8 dims

Many interesting geometries in dim 8 are characterized by 4-forms: elements of $\Lambda^4(\mathbb{R}^8)^*$, the isotropy representation of the symmetric space $E^7/SU(8)$. The complicated orbit structure for $SL(8,\mathbb{R})$ acting on the 4-forms can be understood via roots of E_7 [V].

We focus on the inclusions

$$\mathrm{Sp}(2) \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \mathrm{Sp}(2)\mathrm{Sp}(1) \\ \\ \mathrm{Spin}\,7 \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \mathrm{SO}(8).$$

$\mathrm{Sp}(2)$ fixes a triple $\omega_1, \omega_2, \omega_3$, whilst

$\mathrm{Sp}(2)\mathrm{Sp}(1)$	stabilizes	$\Omega = \omega_1\wedge\omega_1 + \omega_2\wedge\omega_2 + \omega_3\wedge\omega_3$
$\mathrm{Spin}\,7$	stabilizes	$\Phi = -\omega_1\wedge\omega_1 + \omega_2\wedge\omega_2 + \omega_3\wedge\omega_3.$

We shall investigate the topology defined by the two rank 3 groups.

Spinors

Proposition. If a compact, oriented M^8 has a $\mathrm{Spin}\,7$ or a $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure then M is spin and $8\chi = 4p_2 - p_1^2$.

Proof.

$$\Delta_+ \cong \begin{cases} \Lambda^0 \oplus \Lambda^2_7 & = 1+7 \quad \text{for } \mathrm{Spin}\,7 \\ S^2H \oplus \Lambda^2_0 E & = 3+5 \quad \text{for } \mathrm{Sp}(2)\mathrm{Sp}(1) \end{cases}$$

and in both cases $\Delta_- \cong \Lambda^1 \cong TM$. The Euler class e satisfies

$$\mathrm{ch}(\Delta_+ - \Delta_-) = e\,\hat{A}^{-1} = e\big(1 - \tfrac{1}{24}p_1 + \hat{A}_2 + \cdots\big)^{-1}$$

where $\hat{A}_2 = \frac{1}{5760}(7p_1^2 - 4p_2)$.

Theorem. If $\mathrm{Hol}(M) \subseteq \mathrm{Sp}(2)Sp(1)$ and $s > 0$ then $\hat{A}_2 = 0$. If $\mathrm{Hol}(M) = \mathrm{Spin}\,7$ then $\hat{A}_2 = 1$.

So the QK 8-manifolds $\mathbb{HP}^2$, $\mathrm{G}_2/SO(4)$, $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ all admit a $\mathrm{Spin}\,7$ structure but not the rival holonomy!

The remarkable space $G_2/SO(4)$

- parametrizes coassociative 4-planes in $\mathbb{R}^7 \subset \mathbb{O}$ [HL].

- As an application, its orbits under $\mathrm{SO}(4)$ are relevant to the classification of coassociative submanifolds of the G_2 manifold $\Lambda^2_-(S^4)$ that are deformations of T^*S^2 [KS].

- Removing $\mathbb{CP}^2$ and quotienting out by $\mathbb{Z}_3$, we get $\mathscr{N}/\mathbb{H}^*$, where $\mathscr{N}$ is the principal nilpotent orbit in $\mathrm{sl}(3,\mathbb{C})$. The latter is HK and the quotient QK [K,S].

- There is a construction of QK metrics from 5-manifolds with generic 2-plane distributions that are modelled asymptotically on the noncompact dual $\mathrm{G}_2^s/\mathrm{SO}(4)$ [L,B,D].

A Dirac complex

If M^8 has a $\mathrm{Spin}\,7$ structure then $(\mathscr{A}^\bullet, D)$ coincides with

$$0 \to \Omega^0 \to \Omega^1 \to \Omega^2_7 \to 0.$$

Only if it is written backwards do we need the holonomy condition!

It is easy to compute the deformation index

$$h^0 - h^1 + h^2 = \int \mathrm{ch}(\mathrm{ad}\, Q)\, \hat{A}$$

though h^2 is again unknown.

Example. For the Levi-Civita instanton Λ^1 on a manifold with holonomy equal to $\mathrm{Spin}\,7$, it equals $8 - \frac{1}{3}\chi$. The latter is an integer because

$$25 + b_2 + 2b_4^- = b_3 + b_4^+.$$

Intrinsic torsion

…of a $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure lies in $\Lambda^1 \otimes \mathfrak{m}$ =

$E\,H$	$K\,H$
$E\,S^3H$	$K\,S^3H$

M^8 is *quaternionic* if $\tau \in$ row 1; it is *ideal* if $\tau \in$ col 1.

Surprise. $\mathrm{SU}(3)$ possesses invariant structures of both types [J,M].

If M^8 is quaternionic, there is an elliptic complex

$$0 \to \Gamma(S^2H) \to \Gamma(EH) \to \Gamma(\Lambda^2_0 E) \to 0,$$

which corresponds to the sheaf $\mathscr{O}(-2)$ on the twistor space Z^5. Tensor by S^2H to obtain $(\mathscr{A}^\bullet, D)$. Passing to $\mathscr{O}(-3)$ gives

$$0 \to \Gamma(H) \xrightarrow{\partial} \Gamma(E) \xrightarrow{\square} \Gamma(E) \longrightarrow \Gamma(H) \to 0$$

where ∂ is a Fueter operator and $\square$ is second order [B].

Some K theory

Associated to the Dirac operator on $\mathbb{HP}^2$ is the virtual vector bundle

$$\Lambda^2_0(E-H) = \Lambda^2_0 E - E\,H + S^2 H.$$

Now, $E-H$ can't be a genuine vector bundle because:

• any monomorphism $H \to E$ would define a nowhere zero section of $E \otimes H \cong T\mathbb{HP}^2$ but $\chi = 3$;

• $E-H$ has rank 2, but a calculation shows $c_4(E-H) \neq 0$!

In fact, $c(H) = 1-h$ so

$$c(E-H) = c(\underline{\mathbb{C}}^6 - 2H) = (1-h)^{-2} = 1 + 2h + 3h^2.$$

By contrast,

$$c(\Lambda^2_0 E - H) = c(\Delta_+ - S^2 H - H) = 1 - 3h.$$

We shall see that this time the difference is a vector bundle.

Horrocks's instanton

Theorem. There exists a rank 3 complex vector bundle V over $\mathbb{HP}^2$ with $c_2 = 3h$, and an $\mathrm{SU}(3)$-connection with $F \in \Gamma(\mathfrak{sp}(2))$.

Proof. Recall that $E = \ker(p_1 \colon \underline{\mathbb{C}}^6 \to H)$. Similarly,

$$\Lambda^2_0 E = \ker\left(\Lambda^2_0(\underline{\mathbb{C}}^6) \longrightarrow \underline{\mathbb{C}}^6 \wedge H\right).$$

Fix a reduction of $\mathrm{Sp}(3)$ to $\mathrm{SU}(3)$, giving $\mathbb{C}^6 = \Lambda^{1,0} \oplus \Lambda^{0,1}$ and

$$p_2 \colon \Lambda^2_0(\underline{\mathbb{C}}^6) \longrightarrow \underline{\mathbb{C}}^6.$$

Then $p_1 \circ p_2$ has rank 2 everywhere. The instanton connection on $V = \ker(p_1 \circ p_2)$ is induced from that on $\Lambda^2_0 E$ and ultimately E.

The moduli space of such instantons is then the total space of

$$\frac{\mathrm{SL}(3,\mathbb{H})}{SU(3)} \longrightarrow \frac{\mathrm{SL}(3,\mathbb{H})}{Sp(3)}, \text{ whose } T_o \cong \Lambda^2_0(\mathbb{C}^6)$$

Summary

- In parallel to the theory of manifolds with reduced holonomy, there is a unified theory of instantons (which arguably preceded it in the exceptional cases).

- The quaternionic version of ADHM has many unanswered questions and still open problems [O]. Unlike for G_2 or $\mathrm{Spin}\,7$, there is no rich theory of submanifolds.

- The link between G_2 and the nearly-Kähler case is striking, especially since it is unknown if there are compact NK manifolds other than the four usual suspects (S^6, $\mathbb{CP}^3$, $\mathbb{F}^3$, $S^3 \times S^3$).

- More applications are needed of the differential complexes, their cohomology and index theory. A big problem is to determine h^2.

- The exceptional cases (NK, G_2, $\mathrm{Spin}\,7$) suffer from being non-holomorphic theories, with no twistor spaces to retire to. But this did not stop algebraic geometry being used in the construction of new compact manifolds with holonomy G_2 [CHNP].