Instantons on the manifolds of Bryant and Salamon

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Overview

1. The group G_2
2. The Bryant-Salamon examples
3. Construction of G_2-instantons on the Bryant-Salamon manifold
4. Asymptotic behaviour of the instantons
5. Rapid discussion of $Spin(7)$ case

Holonomy and the group G_2

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- What is the group G_2?

$$\begin{aligned} \text{Recall}: \quad \mathbb{O} &= \text{octonians} \\ &= \text{ 8-dim. non-assoc. algebra with 1} \\ &\sim \mathbb{H} + \mathbb{H}\varepsilon. \end{aligned}$$

The group G_2

Definition

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\begin{aligned}
G_2 &= \mathrm{Aut}(\mathbb{O}) \\
&= \{g \in GL(8,\mathbb{R}) \; ; \; g(xy) = g(x)g(y), \forall x, y \in \mathbb{O}\}.
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$$\begin{aligned} G_2 &= \{g \in GL(7,\mathbb{R})\ ;\ g^*\phi = \phi\} \\ \phi &= dx^{123} + dx^1 \wedge (dx^{45} - dx^{67}) \\ &\quad + dx^2 \wedge (dx^{46} - dx^{75}) + dx^3 \wedge (dx^{47} - dx^{56}). \end{aligned}$$

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- By wedge product and contraction, the form ϕ algebraically determines the euclidean metric on $\mathbb{R}^7$.

The group G_2

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Examples of (M, φ) **with** $\mathrm{Hol}(g_\varphi) = G_2$:

- Bryant ('85) - local, incomplete,
- Bryant-Salamon ('89) - complete, noncompact,
- Joyce ('95) - compact,
- Kovalev ('00) - compact.

Manifolds of Bryant and Salamon

Metrics of holonomy $Hol(g_\varphi) = G_2$ on

- $M_1 = \mathcal{S}(S^3)$,
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Also, metric of holonomy $Hol(g_\Phi) = Spin(7)$ on

- $M_4 = \mathcal{S}^-(S^4)$.

Also discovered by Gibbons, Page, Pope in the study of bundle constructions of Ricci-flat metrics.

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$\mathcal{F}$ admits forms

- ω - canonical $\mathbb{R}^3$-valued "soldering" form
- ϕ - Levi-Civita conn. 1-form.

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a rank 4 real vector bundle.

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Total space of $\mathcal{S}$ is non-compact and 7 dimensional. The product $\widetilde{\mathcal{F}} \times \mathbb{H}$ admits

$$\begin{aligned} a &: \widetilde{\mathcal{F}} \times \mathbb{H} \to \mathbb{H} \quad \text{projection} \\ \alpha &= da - a\phi, \quad \mathbb{H} - \text{valued } 1 - \text{form on } \widetilde{\mathcal{F}} \times \mathbb{H}. \end{aligned}$$

Bryant-Salamon construction

Consider the forms

$$\begin{aligned}\gamma_1 &= \omega^{123}\\ \gamma_2 &= \omega^1 \wedge (\alpha^{01} - \alpha^{23}) + \omega^2 \wedge (\alpha^{02} - \alpha^{31}) + \omega^3 \wedge (\alpha^{03} - \alpha^{12}).\end{aligned}$$

ω and α are equivariant, defined on $\widetilde{\mathcal{F}} \times \mathbb{H}$, but γ_1, γ_2 are invariant and descend to $\mathcal{S}$.

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Theorem (Bryant-Salamon)

For the functions

$$f(r) = (1+r)^{1/3} \qquad g(r) = 2(1+r)^{-1/6}$$

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the form φ satisfies $d\varphi = 0$ and $d^{*_\varphi}\varphi = 0$. Furthermore, $Hol(g_\varphi) = G_2$.

G_2-instantons

Recall : Self-dual Yang-Mills equations in 4-dimensions : $*F_A = F_A$.

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On a G_2-manifold $(M^7, \varphi, g_\varphi)$, have decomposition :

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Definition

A G_2-**instanton** is a connection A on a vector bundle $E \to M$ that satisfies (any of) :

- $F_A \in \Lambda^2_{14} \otimes \mathfrak{g}_E$
- $(*\varphi) \wedge F_A = 0$
- $*(\varphi \wedge F_A) = F_A$.

Related constructions

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- Monopoles on $\Lambda^+(S^4), \Lambda^+(\mathbb{CP}^2)$ constructed by Oliveira,
- Instantons on Nearly-Kähler manifolds and cones over them studied by Harland, Nölle, Ivanova, Lechtenfeld, Popov and collaborators,

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and

$$(*\varphi) \wedge F_A = \left(\tau(f' + f^2) - \frac{\sigma}{4} f\right) \Phi$$

where $\sigma = \frac{16}{(1+r)^{2/3}}$ and $\tau = -12(1+r)^{1/3}$ are the coefficient functions of $*\varphi$ from Bryant-Salamon.

Construction of instantons

$A = fA_1$ satisfies $*\varphi \wedge F_A = 0$ if and only if

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Theorem

For the function

$$f(r) = \frac{2}{3(r+1) + C(r+1)^{1/3}},$$

$A = f(r) Im(a\bar{\alpha})$ *defines a* G_2*-instanton on a trivial* $\mathbb{H}$*-bundle over* $\mathcal{S}$.

Asymptotic behaviour

S^3 has symmetric metric : $S^3 = (S^3 \times S^3)/S^3$, and

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$$g_\gamma = \frac{1}{1-(\frac{3}{\rho})^3} d\rho^2 + \frac{4}{9}\rho^2 \left(1 - \left(\frac{3}{\rho}\right)^3\right)(\theta_3 - \phi)^2 + \frac{1}{3}\rho^2\omega^2$$

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Asymptotic, as $\rho \to \infty$, to the cone metric,

$$g_{con} = d\rho^2 + \rho^2\left(\frac{4}{9}(\theta_3-\phi)^2 + \frac{1}{3}\omega^2\right).$$

Ie, cone over the *Nearly Kähler* metric on $S^3 \times S^3$.

Asymptotic behaviour

Definition

A *Nearly Kähler* 6-manifold is an almost-Hermitian manifold (M^6, J, ϖ) with $\Omega = \Omega_1 + i\Omega_2 \in \Omega^{3,0}(M)$ satisfying

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Definition

$E \to M$ a bundle on a NK 6-manifold. A connection A is *Hermitian-Yang-Mills* if the curvature satisfies

$$\begin{aligned} F_A \wedge \varpi^2 &= 0, \\ F_A \wedge \Omega &= 0. \end{aligned}$$

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- Same definition as in Kähler case, that $F_A \in \Lambda_0^{1,1}$ is primitive.
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The differential form A_1 satisfies

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$$A = f(r)A_1 = \frac{-2\left(\left(\frac{\rho}{3}\right)^3 - 1\right)}{\frac{1}{9}\rho^3 + \frac{C}{3}\rho} g_3(\theta_3 - \phi)g^{-1}$$

for $\rho = 3(1 + r^2)^{1/3}$.

Connection at infinity

Theorem

1. *As $\rho \to \infty$, the connection A converges to*

$$\widetilde{A} = \frac{-2}{3} g_3(\theta_3 - \phi) g_3^{-1}.$$

   - *Connection on trivial bundle over $S^3 \times S^3$, pulled back to cone $\mathbb{R}^+ \times S^3 \times S^3$.*

2. *$\widetilde{A}$ is Hermitian-Yang-Mills connection on $S^3 \times S^3$.*

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-

$$\mathcal{F} \times \mathbb{H} \to X$$

is a non-trivial principal $SU(2)$-bundle over X, with connection form ϕ (half LC connection from S^4).

Results for *Spin*(7)

- Consider connections of the form $A = \phi + f(r)A_2$ for $A_2 = \mathrm{Im}(\bar{a}\alpha)$ where $f(r)$ is a function of the radial direction in the $\mathbb{H}$-fibres,

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Theorem

For the function

$$f(r) = \frac{1}{r(1 + D(1+r)^{3/5})} + \frac{D(2r+5)}{5r(1+r)^{2/5}(1 + D(1+r)^{3/5})}$$

$A = \phi + f(r)A_2$ *defines a Spin*(7)*-instanton on bundle* $\mathcal{F} \times \mathbb{H}$ *over* X.