Instantons on the manifolds of Bryant and Salamon

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- 2 The Bryant-Salamon examples
- 3 Construction of G_2 -instantons on the Bryant-Salamon manifold
- 4 Asymptotic behaviour of the instantons
- 5 Rapid discussion of Spin(7) case

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- What is the group *G*₂?

Recall :
$$\mathbb{O}$$
 = octonians
= 8-dim. non-assoc. algebra with 1
 $\sim \mathbb{H} + \mathbb{H}\varepsilon$.

Definition

$$G_2 = \operatorname{Aut}(\mathbb{O})$$

= {g \in GL(8, \mathbb{R}) ; g(xy) = g(x)g(y), \forall x, y \in \mathbb{O}}.

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• G_2 preserves 1, so preserves complement $\langle 1 \rangle^{\perp} \sim \mathbb{R}^7$.

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- There exists non-degenerate 3-form $\phi \in \Lambda^3 \mathbb{R}^7$ such that

$$G_2 = \{g \in GL(7,\mathbb{R}) ; g^*\phi = \phi\}$$

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$$\begin{array}{rcl} G_2 &=& \{g \in GL(7,\mathbb{R}) \; ; \; g^*\phi = \phi\} \\ \phi &=& dx^{123} + dx^1 \wedge (dx^{45} - dx^{67}) \\ && + dx^2 \wedge (dx^{46} - dx^{75}) + dx^3 \wedge (dx^{47} - dx^{56}). \end{array}$$

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 By wedge product and contraction, the form φ algebraically determines the euclidean metric on R⁷.

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G₂-Instantons

For (M^7, g, ∇) riemannian, $\exists \varphi \in \Omega^3(M)$ pointwise isomorphic to ϕ , s.t. $\nabla \varphi = 0 \iff \operatorname{Hol}(g) \subseteq G_2$.

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Then $\operatorname{Hol}(g_{\varphi}) \subseteq G_2$.

Examples of (M, φ) with $Hol(g_{\varphi}) = G_2$:

- Bryant ('85) local, incomplete,
- Bryant-Salamon ('89) complete, noncompact,
- Joyce ('95) compact,
- Kovalev ('00) compact.

Metrics of holonomy $Hol(g_{\varphi}) = G_2$ on

•
$$M_1 = \mathcal{S}(S^3)$$
,

•
$$M_2 = \Lambda^+(S^4)$$
,

•
$$M_3 = \Lambda^+(\mathbb{CP}^2)$$

Image: A matrix and a matrix

Metrics of holonomy $Hol(g_{\varphi}) = G_2$ on

- $M_1 = S(S^3)$,
- $M_2 = \Lambda^+(S^4)$,
- $M_3 = \Lambda^+(\mathbb{CP}^2)$

Also, metric of holonomy $Hol(g_{\Phi}) = Spin(7)$ on

• $M_4 = S^-(S^4)$.

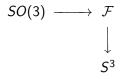
Also discovered by Gibbons, Page, Pope in the study of bundle constructions of Ricci-flat metrics.

Construction of Bryant-Salamon

- complete metric with $Hol(g_{arphi}) = G_2$
- bundle construction
- on $\mathcal{S}(S^3)$, total space of spinor bundle over S^3 .

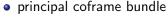
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 - principal coframe bundle
 - set of all $u: T_xS^3 \to \mathbb{R}^3$ for all $x \in S^3$

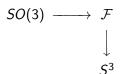


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- set of all $u: T_xS^3 \to \mathbb{R}^3$ for all $x \in S^3$
- ${\mathcal F}$ admits forms
 - ω canonical \mathbb{R}^3 -valued "soldering" form
 - ϕ Levi-Civita conn. 1-form.



Consider the spin double cover,

$$\begin{array}{ccc} SU(2) & \longrightarrow & \widetilde{\mathcal{F}} \\ & & \downarrow \\ & & \varsigma^3 \end{array}$$

Consider the spin double cover, and associated vector bundle

$$SU(2) \longrightarrow \widetilde{\mathcal{F}}$$

$$\downarrow$$
 S^3

$$\mathcal{S} = (\widetilde{\mathcal{F}} \times \mathbb{H})/SU(2),$$

a rank 4 real vector bundle.

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 a rank 4 real vector bundle.

Total space of S is non-compact and 7 dimensional. The product $\widetilde{\mathcal{F}} \times \mathbb{H}$ admits

$$egin{array}{rcl} m{a} & : & \widetilde{\mathcal{F}} imes \mathbb{H} o \mathbb{H} & ext{projection} \ lpha & = & m{d}m{a} - m{a}\phi, & \mathbb{H} - ext{valued} \ 1 - ext{form on} & \widetilde{\mathcal{F}} imes \mathbb{H}. \end{array}$$

$$\begin{array}{rcl} \gamma_1 & = & \omega^{123} \\ \gamma_2 & = & \omega^1 \wedge (\alpha^{01} - \alpha^{23}) + \omega^2 \wedge (\alpha^{02} - \alpha^{31}) + \omega^3 \wedge (\alpha^{03} - \alpha^{12}). \end{array}$$

 ω and α are equivariant, defined on $\widetilde{\mathcal{F}} \times \mathbb{H}$, but γ_1, γ_2 are invariant and descend to \mathcal{S} .

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Theorem (Bryant-Salamon)

For the functions

$$f(r) = (1+r)^{1/3}$$
 $g(r) = 2(1+r)^{-1/6}$

the form φ satisfies $d\varphi = 0$ and $d^{*_{\varphi}}\varphi = 0$.

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the form φ satisfies $d\varphi = 0$ and $d^{*_{\varphi}}\varphi = 0$. Furthermore, $Hol(g_{\varphi}) = G_2$.

Recall : Self-dual Yang-Mills equations in 4-dimensions : $*F_A = F_A$.

Image: Image:

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$$A = rac{\mathsf{Im}(xdar{x})}{1+|x|^2}$$
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On a G_2 -manifold $(M^7, \varphi, g_{\varphi})$, have decomposition :

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For example : $\Lambda^2_{14} = \ker \{ * \varphi \land \cdot : \Lambda^2 \to \Lambda^6 \}.$

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Definition

A G_2 -instanton is a connection A on a vector bundle $E \rightarrow M$ that satisfies (any of) :

- $F_A \in \Lambda^2_{14} \otimes \mathfrak{g}_E$
- $(*\varphi) \wedge F_A = 0$

•
$$*(\varphi \wedge F_A) = F_A.$$

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- Monopoles on $\Lambda^+(S^4), \Lambda^+(\mathbb{CP}^2)$ constructed by Oliveira,
- Instantons on Nearly-Kähler manifolds and cones over them studied by Harland, Nölle, Ivanova, Lechtenfeld, Popov and collaborators,

Construction of Instantons

Consider the form $\mathfrak{su}(2)$ -valued form $A_1 = \operatorname{Im}(a\bar{\alpha})$.

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= $(rf' + 2f - rf^{2}) \frac{\alpha \wedge \bar{\alpha}}{2} - (f' + f^{2}) \frac{a\bar{\alpha} \wedge \alpha\bar{a}}{2} - \frac{1}{2}a\Omega\bar{a}.$

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and

$$(*\varphi) \wedge F_{\mathcal{A}} = \left(\tau(f'+f^2)-\frac{\sigma}{4}f\right)\Phi$$

where $\sigma = \frac{16}{(1+r)^{2/3}}$ and $\tau = -12(1+r)^{1/3}$ are the coefficient functions of $*\varphi$ from Bryant-Salamon.

 $A = fA_1$ satisfies $*\varphi \wedge F_A = 0$ if and only if

$$f' + f^2 + \frac{1}{3} \frac{1}{r+1} f = 0.$$

Riccati-type equation, can be solved explicitly.

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Theorem

For the function

$$f(r) = \frac{2}{3(r+1) + C(r+1)^{1/3}},$$

 $A = f(r)Im(a\bar{\alpha})$ defines a G_2 -instanton on a trivial \mathbb{H} -bundle over S.

S^3 has symmetric metric : $S^3=(S^3\times S^3)/S^3,$ and $\mathcal{S}=(S^3\times S^3\times \mathbb{H})/S^3$

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Here, ω, ϕ can be written in terms of the Maurer-Cartan forms $\theta_1, \theta_2, \theta_3$ from the three S^3 factors.

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$$g_{\gamma} = rac{1}{1-(rac{3}{
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ho^2 + rac{4}{9}
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Asymptotic, as $\rho \to \infty$, to the cone metric,

$$g_{con}=d
ho^2+
ho^2\left(rac{4}{9}(heta_3-\phi)^2+rac{1}{3}\omega^2
ight).$$

Ie, cone over the *Nearly Kähler* metric on $S^3 imes S^3_{
m cm}$,

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Definition

A Nearly Kähler 6-manifold is an almost-Hermitian manifold (M^6, J, ϖ) with $\Omega = \Omega_1 + i\Omega_2 \in \Omega^{3,0}(M)$ satisfying

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 $(S^3 \times S^3, g_{nk})$ is Nearly Kähler with $S^3 \times S^3 \times S^3$ -symmetry. Ω and ϖ defined in terms of ω, ϕ, θ_3 , etc.

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Definition

 $E \rightarrow M$ a bundle on a *NK* 6-manifold. A connection *A* is *Hermitian-Yang-Mills* if the curvature satisfies

$$\begin{array}{rcl} F_A \wedge \varpi^2 &=& 0, \\ F_A \wedge \Omega &=& 0. \end{array}$$

- Same definition as in Kähler case, that $F_A \in \Lambda_0^{1,1}$ is primitive.
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The differential form A_1 satisfies

$$A_1 = \mathsf{Im}(a\bar{\alpha}) = -r^2 g_3(\theta_3 - \phi) g_3^{-1}$$

(well-defined on $(S^3 imes S^3 imes S^3)/S^3)$

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$$A = f(r)A_1 = \frac{-2\left(\left(\frac{\rho}{3}\right)^3 - 1\right)}{\frac{1}{9}\rho^3 + \frac{C}{3}\rho}g_3(\theta_3 - \phi)g^{-1}$$

for $\rho = 3(1 + r^2)^{1/3}$.

Theorem

$${f 0}$$
 As $ho
ightarrow \infty$, the connection A converges to

$$\widetilde{A}=\frac{-2}{3}g_3(\theta_3-\phi)g_3^{-1}.$$

• Connection on trivial bundle over $S^3 \times S^3$, pulled back to cone $\mathbb{R}^+ \times S^3 \times S^3$.

2 \widetilde{A} is Hermitian-Yang-Mills connection on $S^3 \times S^3$.

• Connection A on $E \to X^8$ is Spin(7)- instanton if $\Phi \wedge F_A = *F_A$,

Image: A matrix and a matrix

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Results for Spin(7)

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- Bryant-Salamon construction :

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$\mathcal{F} \times \mathbb{H} \to X$

is a non-trivial principal SU(2)-bundle over X, with connection form ϕ (half LC connection from S^4).

Consider connections of the form A = φ + f(r)A₂ for A₂ = Im(āα) where f(r) is a function of the radial direction in the H-fibres,

• Consider connections of the form $A = \phi + f(r)A_2$ for $A_2 = \text{Im}(\bar{a}\alpha)$ where f(r) is a function of the radial direction in the \mathbb{H} -fibres,

Theorem

For the function

$$f(r) = \frac{1}{r(1+D(1+r)^{3/5})} + \frac{D(2r+5)}{5r(1+r)^{2/5}(1+D(1+r)^{3/5})}$$

 $A = \phi + f(r)A_2$ defines a Spin(7)-instanton on bundle $\mathcal{F} \times \mathbb{H}$ over X.