Determinantal Singularities

– Abstracts –

Workshop at the Riemann Center,
Leibniz Universität Hannover,
March 5–9, 2018

Schedule

	Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
9:15 – 10:15	V. Goryunov	W. Ebeling	T. Gaffney	M. Ruas	H. Pedersen
10:45 – 11:45	N. Grulha	A. Pichon	M. Tosun	D. Kerner	G. Penafort
12:00 – 13:00	*lunch*	*lunch*	*lunch*	*lunch*	*lunch*
13:00 – 15:00	*discussion*	*discussion*	*discussion*	*discussion*	*discussion*
15:00 – 16:00	X. Zhang	*Poster session*	*free afternoon*	J. Stevens	*end*
16:30 – 17:30	J. Damon	D. Siersma		M. Pereira	

Note: On Monday we intend to start the first talk at 9:30 instead of 9:15.

Contents

James Damon: University of North Carolina, United States

Schubert Decomposition and Topology of Milnor Fibers of the Varieties of Singular Matrices

We investigate the topology of the varieties of singular $m \times m$ complex matrices which may be either general, symmetric or skew-symmetric (with m even). Although these hypersurfaces have highly nonisolated singularities, we determine the topology of their Milnor fibers (along with the complements and links). We do so by replacing the Milnor fibers by corresponding global Milnor fibers which have a "complex geometry" resulting from a transitive action of an appropriate linear algebraic group. This yields compact "model submanifolds" for the homotopy types of the Milnor fibers which are classical symmetric spaces in the sense of Cartan. Unlike isolated singularities, the cohomology of the Milnor fibers (and complements) are isomorphic as algebras to exterior algebras or for one family, modules over exterior algebras; and cohomology of the link is, as a vector space, a truncated and shifted exterior algebra

Second, we give a Schubert cell decomposition of the global Milnor fibers by using the Cartan model for these symmetric spaces together with Iwasawa decomposition combined with results due to a number of authors concerning the Schubert decomposition of Lie groups and symmetric spaces. The "Schubert cycles", which are the closures of the Schubert cells, are images of products of suspensions of projective spaces (complex, real, or quaternionic as appropriate). In the cases of general or skew-symmetric matrices the Schubert cycles have fundamental classes, and for symmetric matrices mod 2 classes, which give a basis for the homology. They are also shown to correspond to the cohomology generators for the symmetric spaces. For general matrices the duals of the Schubert cycles are represented as explicit monomials in the generators of the exterior cohomology algebra; and for symmetric matrices they are related to Stiefel-Whitney classes of an associatedvector bundle.

The pull-backs of these cohomology classes generate a characteristic subalgebra of the cohomology of the Milnor fibers of the matrix singularities of these types.

These results extend to exceptional orbit hypersurfaces, complements, and links.

Wolfgang Ebeling, Leibniz Universität Hannover, Germany

Indices of 1-forms on essentially isolated determinantal singularities

We introduce the notion of an essentially isolated determinantal singularity (EIDS). We show how to define analogs of the Poincaré-Hopf index for a 1-form on such a singularity and describe relations between these indices and the radial index. This is joint work with Sabir Gusein-Zade.

Terence Gaffney, Northeastern University, United States

The Landscape of Symmetric Determinantal Singularities and Whitney Equisingularity

(Joint work with Michelle Lira dos Santos Molino)

In this talk we show how the study of the Whitney equisingularity of symmetric determinantal singularities leads naturally into the study of the landscape of determinantal singularities. This study reveals the invariants which control equisingularity. In the case in which the singularity X is defined by the minors of size $n-2$ of a symmetric $n \times n$ matrix F, we show how to compute the invariants which arise from the study of the landscape in terms of the presentation matrix F.

Victor Goryunov, University of Liverpool, United Kingdom

Simple boundary function singularities, symmetric matrices

In our paper from 15 years ago, Zakalyukin and myself related the Bruce-Tari classification of simple symmetric matrices depending on two parameters to certain subgroups X of the Weyl groups A_μ, D_μ, E_μ. These subgroups arise as monodromy groups of the determinantal curves, and their Dynkin diagrams are singled out by a straightforward rule as the subdiagrams of the affine ADE Dynkin diagrams. Bases of miniversal deformations of the matrix singularities turn out to be isomorphic to the quotients of the ADE configuration spaces by such subgroups X.

In the talk, I will consider similar constructions for the boundary version of the Bruce-Tari classification.

Nivaldo de Goes Grulha Junior, ICMC Universidade de São Paulo, Brazil

Chern-Schwartz-MacPherson Classes of Generic Determinantal Varieties

(Joint with Terence Gaffney and Maria A.S. Ruas)

In [4] MacPherson proved the existence and uniqueness of Chern classes for possibly singular complex algebraic varieties. The local Euler obstruction, defined by MacPherson in that paper, was one of the main ingredients in his proof.

The computation of the local Euler obstruction is not easy; various authors propose formulas which make the computation easier. For instance, Lê and Teissier provide a formula in terms of polar multiplicities [3].

In [1], Brasselet, Lê and Seade give a Lefschetz type formula for the local Euler obstruction. The formula shows that the local Euler obstruction, as a constructible function, satisfies the Euler condition relative to generic linear forms.

In order to understand these ideas better, some authors worked on some more specific situations. For example, in the special case of toric surfaces, an interesting formula for the Euler obstruction was proved by Gonzalez–Sprinberg [2], this formula was generalized by Matsui and Takeuchi for normal toric varieties [5].

A natural class of singular varieties to investigate the local Euler obstruction and the generalizations of the characteristic classes is the class of generic determinantal varieties. Roughly speaking, generic determinantal varieties are sets of matrices with a given upper bound on their ranks. Their significance comes, for instance, from the fact that many examples in algebraic geometry are of this type, such as the Segre embedding of a product of two projective spaces. Independently, in recent work [6], Zhang computed the Chern-Mather-MacPherson Class of projectivized determinantal varieties, in terms of the trace of certain matrices associated with the push forward of the MacPherson-Schwartz class of the Tjurina transform of the singularity.

We prove a surprising formula that allow us to compute the local Euler obstruction of generic determinantal varieties using only Newton binomials. Using this formula we also compute the Chern–Schwartz–MacPherson classes of such varieties.

References

[1] J. -P. Brasselet, Lê D. T. and J. Seade, *Euler obstruction and indices of vector fields.* Topology 39 (2000), no. 6, 1193–1208.

[2] G. Gonzalez-Sprinberg, *Calcul de l'invariant local d'Euler pour les singularités quotient de surfaces*, *C. R. Acad. Sci. Paris* Ser. A–B 288, A989–A992 (1979).

[3] Lê D. T. and B. Teissier, *Variétés polaires Locales et classes de Chern des variétés singulières*, *Ann. of Math.* 114, (1981), 457–491.

[4] R. MacPherson, *Chern classes for singular algebraic varieties*, *Ann. of Math.* 100 (1974), 423–432.

[5] Y. Matsui and K. Takeuchi, *A geometric degree formula for A-discriminants and Euler obstructions of toric varieties*, *Adv. Math.* 226, 2040–2064 (2011).

[6] X. Zhang, *Chern-Schwartz-MacPherson Class of Determinantal Varieties*, `arXiv:1605.05380 [math.AG]`.

Dmitry Kerner, Ben Gurion University of the Negev, Israel:

Group actions on modules, large orbits, and finite determinacy

Let M be a filtered module over a ring k, with an action $G \circlearrowright M$. One way to estimate how large is the orbit Gz of an element $z \in M$ is by checking large submodules of M lying inside Gz. We provide the corresponding (necessary/sufficient) conditions in terms of the tangent module $T_{(G,1)}(z)$ and the filtration of M.

This question originates from the classical finite determinacy problems of Singularity Theory, there the ring k was usually the field of complex/real numbers and M was the space of map-germs. We suggest a very general (characteristic free) approach. This extends both the classical criteria of Mather/Tougeron (and many others) and some recent results (in zero/positive characteristics) to a broad class of rings, modules and group actions.

Such a "linearization", i.e. reduction of determinacy questions to the tangent level, is not yet a complete solution. In many cases, e.g. for group actions on matrices, the tangent module is complicated. We address some most important actions and formulate the determinacy criteria in terms of the classical matrix invariants.

Helge Møller Pedersen, Universidade de Fortaleza, Brazil:

Tjurina transform and resolutions of determinantal singularities

Transformation (or modifications) plays an important role in the study of singular varieties. There is the famous result by Hironaka that normalized blow-ups can be used to resolve singularities in characteristic 0, and also Spivakovskys result that normalized Nash transform can be used to resolve singular complex surfaces. In this talk we will focus on the Trurina transform, which is a transform of determinantal singularities defined by their linear structure. We will first discuss the Tjurina transform and its transpose for model determinantal singularities. See how they are related and how the are related to the Nash transform, and determine their homotopy type. We will then define the Tjurina transform (and its transpose) for general determinantal singulaities, show some basic properties and discuss how it differs from the model case. We will also discuss how in many cases the Tjurina transform can easely be computed, and show that often the Tjurina transform (or its transpose) is a complete intersection. Finally we will use the Tjurina transform to resolve some hypersurface singularities.

Guillermo Peñafort Sanchis, BCAM, Bilbao, Spain

What are multiple points?

I will discuss different definitions of multiple point spaces of singular maps and review what each of them is good for.

Miriam da Silva Pereira, Universidade Federal de Paraiba, Brazil:

The bi-Lipschitz equisingularity of essentially isolated determinantal singularities

The bi-Lipschitz geometry is one of the main subjects in the modern approach of Singularity Theory. However, it rises from works of important mathematicians of the last century, especially Zariski. In this work we investigate the Bi-Lipschitz equisingularity of families of Essentially Isolated Determinantal Singularities inspired by the approach of Mostowski and Gaffney.

Anne Pichon, Université de Aix Marseille:

On surface singularities which are Lipschitz normally embedded

Any germ of a complex analytic space is equipped with two natural metrics: the outer metric induced by the hermitian metric of the ambient space and the inner metric, which is the associated riemannian metric on the germ. A germ is called Lipschitz normally embedded if the outer and inner metrics are bilipschitz equivalent. The Lipschitz normal embedding of germs is a remarkable metric property whose study is a longstanding problem. We will present two large classes of surface singularities which are Lipschitz normally embedding. The first one is the class of minimal surface singularities : any minimal surface singularity is Lipschitz normally embedded, and they are the only rational surface singularities with this property. I will explain how this result is closely related with the resolution of singularities. This result is a joint work with Walter Neumann and Helge Pedersen. The second class of examples is among superisolated singularities. This result is a joint work with Filip Misev.

Maria Aparecida Soares Ruas, Universidade de São Paulo, São Carlos, Brazil:

Lipschitz Normal Embeddings in the Space of Matrices

(Joint with Dmitry Kerner and Helge Møller Pedersen)

The aim of the talk is to prove Lipschitz normal embeddedness of some algebraic subsets of the space of matrices. These include the space of rectangular/(skew-)symmetric/hermitian matrices of rank equal to a given number and their closures, and the upper triangular matrices with determinant 0. A short discussion about generalizing these results to determinantal varieties in real and complex spaces will also be made. If time allows, I will present an introduction to Lipschitz geometry of determinantal arrangements.

Dirk Siersma, Universiteit Utrecht, Netherlands:

Polar degree and Huh's conjecture

For any projective hypersurface $V := \{f = 0\} \subset \mathbb{P}^n$, the notion of polar degree is defined as the topological degree of the (projectivized) gradient mapping of the homogeneous polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$:

$$\operatorname{grad} f : \mathbb{P}^n \setminus \operatorname{Sing} V \to \mathbb{P}^n.$$

Thus $\operatorname{pol}(V) := \#(\operatorname{grad} f^{-1}(a)$ for any generic point $a \in \mathbb{P}^n$. The case that $\operatorname{pol}(V) = 0$ is related to the question what happens if the Hessian of f is identically zero. This was solved by Gordan and Noether in 1876. Dolgachev considered in 2000 this invariant under the reason that the gradient mapping is bi-rational if and only if $\operatorname{pol}(V) = 1$. These hypersurfaces are called homaloidal. He classified the projective plane curves which have this property. Huh determined in 2014 all homoloidal hypersurfaces in $\mathbb{P}^n$ with at most isolated singularities. Besides this many non-isolated cases are known, such as the determinant map and relative invariants of pre-homogeneous spaces. In this talk we consider the case $\operatorname{pol}(V) \geq 2$ and proof in the case of surfaces the conjecture of Huh that his list of polar degree 2 surfaces with isolated singularities is complete!
Reference: J. Huh, Milnor numbers of projective hypersurfaces with isolated singularities. Duke Math. J. 163 (2014), no. 8, 1525-1548.

Finally we say something more about the non-isoaled cases.

Jan Stevens, Göteborgs universitet, Sweden:

Formats

Codimension two Cohen-Macaulay singularities are determinantal and so are all their deformations. This is an example of a format. In a naive interpretation a format is a way of writing or coding (efficiently) the equations of a singularity. For rational surface singularities of higher codimension also other formats occur. The determinantal format occurs for the Artin component of the versal deformation, with the Tjurina modification giving the simultaneous RDP-resolution. Its generalisation is the blow-up of a canonical ideal. For Gorenstein 3-fold singularities the Pfaffian format takes the place of the determinantal format.

Meral Tosun, Galatasaray Üniversiti, Turkey:

Root systems, Linear Free Divisors, and Quasi Determinantal Singularities

We will construct geometrically a new root system for certain surface singularities, called quasi determinantal, and we will discuss on a possible construction of a Lie algebra corresponding to our singularities showing that the linear free divisors defined by the new roots are of the special ones.

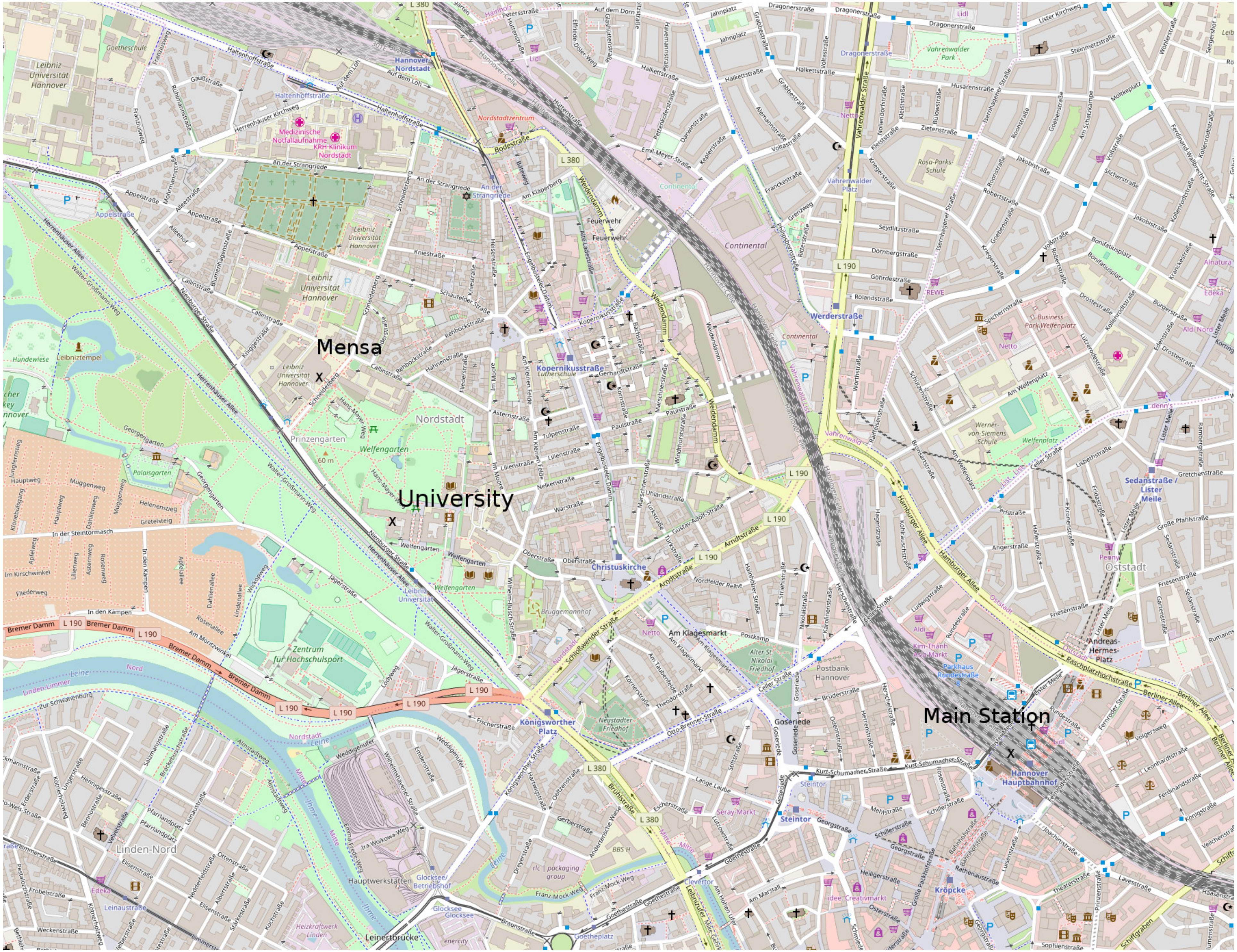
Xiping Zhang, Florida State University, United States:

Local Euler Obstruction and Characteristic Classes of Determinantal Varieties

The (generic) determinantal variety is the projective variety consisting of m by n matrices with kernel dimension $\geq k$, which arises naturally from many aspects. In this talk I will give formulas for the local Euler obstructions on them. With the formulas I will prove that characteristic cycle of the intersection cohomology sheaf of a determinantal variety agrees with its conormal cycle, and hence is irreducible. This is an interesting and rare phenomenon, and has been studied for many spaces. I will also give formulas for the Chern-Mather/Chern-Schwartz-MacPherson classes of determinantal varieties. Knowing the characteristic classes is equivalent to knowing some interesting invariants, such as sectional Euler characteristic and polar degrees. The formulas are based on calculations of degrees of certain Chern classes of the universal bundles over Grassmannians. For low dimensions we use Macaulay2 to exhibit some examples, and formulate conjectures concerning the positivity of the the Chern-Schwartz-MacPherson classes.

Additional information

- All lectures will take place in the room B302 the main building of the university, Welfengarten 1, 30167 Hannover.
- We intentionally left ample room for discussions. There are two rooms available for this purpose: G117 and G123. You may also use G116 as a place to work and have a tea or a coffee.
- There are several options for lunch such as the stundent restaurant (Mensa), the cafeteria inside the main building (Sprengelstube), and various restaurants around (Spandau, Francesca & Fratelli, Pizza Oma, Extrakt, . . .).
- We plan to go for dinner on Tuesday evening, 19:00 at Zwischenzeit. You will need to pay for the dinner yourself. If you wish to join, please let us know at the beginning of the workshop.
- On the free afternoon we will be offering to go to the Herrenhäuser Gärten.
- If you have any further questions, do not hesitate to contact us by email mazach@uni-mainz.de or fruehbis-krueger@math.uni-hannover.de.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